UPPER BOUNDS ON THE RUNNING TIME OF BOOTSTRAP PERCOLATION

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Abstract. For k-graphs F and H_0 the F-bootstrap percolation process (or F-process) starting with H_0 is a sequence $(H_i)_{i\geqslant 0}$ of k-graphs such that H_{i+1} is obtained from H_i by adding all those $e \in V(H_0)^{(k)} \smallsetminus E(H_i)$ as edges that complete a new copy of F. The running time of this F-process, denoted by $M_F(H_0)$, is the smallest i with $H_i = H_{i+1}$. Bollobás proposed the problem of determining the maximum running time for $n \in \mathbb{N}$, i.e., $M_F(n) = \max_{|V(H_0)|=n} M_F(H_0)$. Although this problem has received a lot of attention recently, until now the best known upper bound for $M_{K_t}(n)$, with $t \geqslant 5$, was the trivial bound $\binom{n}{2}$.

Here we provide the first non-trivial upper bound for this problem by showing that

$$M_{K_t}(n) \leqslant \Big(\frac{t-3}{t-2} + o(1)\Big)\binom{n}{2}$$

holds for every integer $t \geqslant 3$. In fact, we prove the following more general result. For every $k \geqslant 2$, every k-graph F, and every $e \in E(F)$ we have $M_F(n) \leqslant \big(\pi(F-e) + o(1)\big)\binom{n}{k}$, where π is the Turán density.

§1 Introduction

A *k-uniform hypergraph* (or *k-graph*) H consists of a vertex set $V(H)$ and a set of edges $E(H) \subseteq V(H)^{(k)}$, where $V(H)^{(k)} = \{S \subseteq V(H) : |S| = k\}$. Given k-graphs F and H, a copy of F in H is a subhypergraph of H that is isomorphic to F. The number of copies of F in H is denoted by $c(F, H)$. The *F-bootstrap percolation process* (or *F-process*) *starting with* H is the sequence $(H_i)_{i\geqslant 0}$ of k-graphs with H_i defined inductively as follows. Set $H_0 = H$ and given H_i let H_{i+1} be the k-graph with vertex set $V(H)$ and edge set

$$E(H_{i+1}) = E(H_i) \cup \left\{e \in V(H)^{(k)} : \ c(F, H_i \cup e) > c(F, H_i)\right\},$$

where $H_i \cup e$ denotes the k-graph $\big(V(H), E(H_i) \cup \{e\}\big)$.

This notion originates in the work of Bollobás [3], where it arose in the context of weak saturation. Since then, connections to other areas such as to cellular automata, more general bootstrap-type dynamics [18], and statistical physics (see [1] and [5]) have been established. Bollobás [4] suggested the problem of determining the maximum running time of such processes (see also [12] for a random variant). More precisely, the running time of the F-process starting with H is $\tau_F(H) = \min\{t \in \mathbb{N}_{\geqslant 0} : \ H_t = H_{t+1}\}$. Given $n \in \mathbb{N}$ (and for $k = 2$), Bollobás asked to determine

$$M_F(n) = \max\left\{\tau_F(H) :\ H \text{ is a } k\text{-graph on } n \text{ vertices}\right\}.$$

One major theme in this area is the maximum running time of K_r-processes, where K_r is the complete graph on r vertices. Bollobás, Przykucki, Riordan, and Sahasrabudhe [4] and, independently, Matzke [17] showed that $M_{K_4}(n) = n-3$, for all $n \geqslant 3$. In [4] the authors also showed that $M_{K_r}(n) \geqslant n^{2-\lambda_r-o(1)}$ for $r \geqslant 5$, where λ_r is some explicit constant with $\lambda_r \to 0$ as $r \to \infty$, and conjectured that for all $r \geqslant 5$, $M_{K_r}(n) = o(n^2)$. Balogh, Kronenberg, Pokrovskiy, and Szabó [2] disproved this conjecture, showing that $M_{K_r}(n) = \Omega(n^2)$ for all $r \geqslant 6$. For $r = 5$, they used Behrend's construction for arithmetic progressions to show that $M_{K_5}(n)$ is asymptotically larger than $n^{2-\varepsilon}$ for all $\varepsilon > 0$. It remains an open problem whether $M_{K_5}(n)$ is quadratic in n or not. Until now, the best known upper bound for $M_{K_t}(n)$, with $t \geqslant 5$, was the trivial upper bound $\binom{n}{2}$. This frustrating state of affairs has been pointed out repeatedly in the literature, for instance in [4], [2], and most recently in the survey [10].

Here we establish the first non-trivial upper bound on $M_{K_t}(n)$ for $t \geqslant 5$.

Theorem 1.1. *For every $t \geqslant 3$ we have $M_{K_t}(n) \leqslant \left(\frac{t-3}{t-2} + o(1)\right)\binom{n}{2}$.*

The $o(1)$ notation in the statement means that for every $\varepsilon > 0$ there is some n_0 such that $M_{K_t}(n) \leqslant \left(\frac{t-3}{t-2} + \varepsilon\right)\binom{n}{2}$ for all $n \in \mathbb{N}$ with $n \geqslant n_0$.

Moreover, we provide a non-trivial upper bound for every graph in terms of the *chromatic number*. Recall that for a graph F the chromatic number $\chi(F)$ denotes the smallest $c \in \mathbb{N}$ such that there is a proper colouring of F with c colours.

Theorem 1.2. *For every graph F and every $e \in E(F)$ we have*

$$M_F(n) \leqslant \left(\frac{\chi(F-e)-2}{\chi(F-e)-1} + o(1)\right)\binom{n}{2}.$$

Note that Theorem 1.2 directly implies Theorem 1.1.

Apart from few exceptions (see the survey by Fabian, Morris, and Szabó [10]), for a general graph F no non-trivial upper bound on $M_F(n)$ was known. One notable exception is the following proposition by Fabian, Morris, and Szabó [9] with a very elegant proof.

Proposition 1.3 ([9])**.** *Let F be a graph with at least two edges. Then $M_F(n) \leqslant 2\mathrm{ex}(n, F)$ for every $n \in \mathbb{N}$.*

Note that this proposition provides non-trivial upper bounds for bipartite graphs F but not for non-bipartite ones. For a further discussion of this proposition and more background on the F-process we refer the reader to the survey by Fabian, Morris, and Szabó [10].

Recently, the investigation of F-processes and their maximum running times was initiated for hypergraphs, see [19], [13], and [8]. While these works established that $M_{K_r^{(k)}}(n) = \Omega(n^k)$ holds for all $k \geqslant 3$ and $r \geqslant k+1$, also in this setting non-trivial upper bounds were lacking. The only non-trivial upper bounds for hypergraphs (of uniformity at least 3) before this work were for $K_4^{(3)-}$, proved in [8], and for extension hypergraphs [16]. In particular, the best known upper bound for any k-uniform clique was so far $\binom{n}{k}$.

We provide the first non-trivial upper bounds in this more general setup as well and indeed Theorem 1.2 is a simple corollary of our general result for hypergraphs. To state it, we briefly recall the following definitions. Given a k-graph F and $n \in \mathbb{N}$, the extremal number of n and F, denoted by $\mathrm{ex}(n, F)$, is the maximum number of edges that a k-graph on n vertices can have without containing a copy of F. The Turán density of a k-graph F is $\pi(F) = \lim_{n\to\infty} \frac{\mathrm{ex}(n,F)}{\binom{n}{k}}$; this limit exists due to a simple monotonicity argument [14]. For the sake of brevity we omit an extensive discussion of this rich area and instead refer the reader to the surveys by Füredi [11], Sidorenko [20], and Keevash [15] for more background.

We obtain the following general upper bound on the maximum running time of F-processes in terms of the Turán density.

Theorem 1.4. *For every integer $k \geqslant 2$, every k-graph F, and every $e \in E(F)$ we have*

$$M_F(n) \leqslant \big(\pi(F-e) + o(1)\big)\binom{n}{k}.$$

From this, Theorem 1.2 follows easily by the Erdős-Stone-Simonovits theorem (see [7] and [6]), which states that for every graph F we have $\pi(F) = \frac{\chi(F)-2}{\chi(F)-1}$.

§2 Preliminaries

Given $i \in \mathbb{N}$, we write $[i] = \{1, \dots, i\}$. For a k-graph H and $e \in V(H)^{(k)}$ denote by $H \cup e$ the k-graph $(V(H), E(H) \cup \{e\})$ and set $v(H) = |V(H)|$. For $e \in E(H)$ we write $H - e$ for the k-graph $\big(V(H), E(H) \setminus \{e\}\big)$. Usually, we omit parentheses around singletons.

We use the following fundamental result, often referred to as supersaturation (see, for instance results in [15, Lemma 2.1 and Theorem 2.2]).

Theorem 2.1 (Supersaturation)**.**
For every k-graph F and $\varepsilon > 0$, there are $\delta > 0$ and n_0 such that every k-graph on $n \geqslant n_0$ vertices with at least $(\pi(F) + \varepsilon)\binom{n}{k}$ edges contains at least $\delta n^{v(F)}$ copies of F.

§3 Proof of Theorem 1.4

The following is a weaker but simpler version of Theorem 1.4.

Theorem 3.1. *For every integer $k \geqslant 2$ and every k-graph F we have*

$$M_F(n) \leqslant \big(\pi(F) + o(1)\big)\binom{n}{k}.$$

Although Theorem 3.1 follows from Theorem 1.4, we give a direct proof here since it illustrates the main idea behind the proof of Theorem 1.4 particularly clearly.

Proof of Theorem 3.1. Given $\varepsilon > 0$, an integer $k \geqslant 2$, and a k-graph F, choose $\delta > 0$ and $n \in \mathbb{N}$ such that

$$\varepsilon, k, v(F) \gg \delta \gg n^{-1}.$$

We need to show that $M_F(n) \leqslant \big(\pi(F) + \varepsilon\big)\binom{n}{k}$. Note that if F has only one edge, the result is trivial, so assume that F has at least two edges. Let $(H_i)_{i\geqslant 0}$ be an F-process starting with $H = (V, E)$, where $|V| = n$, that satisfies $M_F(n) = \tau_F(H) =: \tau$. For each $i \in [\tau]$ choose $e_i \in E(H_i) \smallsetminus E(H_{i-1})$ arbitrarily. Let G be the graph on V whose edge set is $\{e_i : i \in [\tau]\}$. Assume that $\tau \geqslant (\pi(F) + \varepsilon)\binom{n}{k}$. We aim to derive a contradiction by establishing upper and lower bounds for the number of copies of F in G. Denote the set of copies of F in G by $\mathcal{F}$.

On the one hand, supersaturation and the hierarchy of constants imply that

$$|\mathcal{F}| \geqslant \delta n^{v(F)}.$$

For any $D \in \mathcal{F}$ let $i(D) = \max\{i \in [\tau] : e_i \in E(D)\}$. Now observe that for a copy D of F in G we must have $e_{i(D)-1} \in E(D)$. Otherwise, by the definition of G, for every edge e_i of $D - e_{i(D)}$ we have $i \leqslant i(D) - 2$ and so $D - e_{i(D)} \subseteq H_{i(D)-2}$. Therefore, the definition of an F-process implies that $e_{i(D)} \in E(H_{i(D)-1})$, a contradiction. Thus, for every $e_j \in E(G)$, with $j \geqslant 2$ and $v \in e_{j-1} \smallsetminus e_j$ we know that v is contained in every copy $D \in \mathcal{F}$ with $e_{i(D)} = e_j$. This entails that there are at least $k + 1$ vertices (namely v and the vertices in e_j) which are contained in every copy $D \in \mathcal{F}$ with $e_{i(D)} = e_j$. Since of course no $D \in \mathcal{F}$ satisfies $i(D) = 1$, we conclude that for every $e \in E(G)$ it holds that $|\{D \in \mathcal{F} : e = e_{i(D)}\}| \leqslant n^{v(F)-k-1}$. Hence, we obtain

$$|\mathcal{F}| = \sum_{e\in E(G)} |\{D \in \mathcal{F} : e = e_{i(D)}\}| \leqslant n^k \cdot n^{v(F)-k-1} = n^{v(F)-1},$$

contradicting the lower bound above because of the hierarchy. □

Proof of Theorem 1.4. Given $\varepsilon > 0$, an integer $k \geqslant 2$, and a k-graph F, choose $\delta > 0$ and $n \in \mathbb{N}$ such that

$$\varepsilon, k, v(F) \gg \delta \gg n^{-1}.$$

Let $e \in E(F)$ and set $F^- = F - e$. We need to show that $M_F(n) \leqslant \big(\pi(F^-) + \varepsilon\big)\binom{n}{k}$. One can observe that the statement holds if F has at most two edges, so assume that F has at least three edges. Let $(H_i)_{i\geqslant 0}$ be an F-process starting with $H = (V, E)$, where $|V| = n$, that satisfies $M_F(n) = \tau_F(H) =: \tau$. For each $i \in [\tau]$ choose $e_i \in E(H_i) \smallsetminus E(H_{i-1})$ arbitrarily. Let G be the graph on V whose edge set is $\{e_i : i \in [\tau]\}$. Assume that $\tau \geqslant (\pi(F^-) + \varepsilon)\binom{n}{k}$. We aim to derive a contradiction by establishing upper and lower bounds for the number of copies of F^- in G. Denote the set of copies of F^- in G by $\mathcal{F}$. Supersaturation and the hierarchy of constants imply that

$$|\mathcal{F}| \geqslant \delta n^{v(F^-)} = \delta n^{v(F)} \,. \tag{3.1}$$

For every $D \in \mathcal{F}$ define $i(D) = \max\{i \in [\tau] : e_i \in E(D)\}$ and $j(D) = \max\{j \in [\tau] : e_j \in E(D) \smallsetminus e_{i(D)}\}$. Furthermore, for every $D \in \mathcal{F}$ fix some $e_D \in V^{(k)}$ such that $D \cup e_D$ is a copy of F (if there are several choices, pick one arbitrarily). For $e \in E(H_\tau)$ let $\mathfrak{s}(e)$ denote the step in which e is added, i.e., $\mathfrak{s}(e) = \min\{s \in [\tau] : e \in E(H_s) \smallsetminus E(H_{s-1})\}$ if $e \notin E(H_0)$ and $\mathfrak{s}(e) = 0$ if $e \in E(H_0)$. Depending on how $\mathfrak{s}(e_D)$ relates to $j(D)$ and $i(D)$, we assign to every $D \in \mathcal{F}$ one of the following four types.

Type 1: $\mathfrak{s}(e_D) \leqslant j(D)$,
Type 2: $j(D) < \mathfrak{s}(e_D) < i(D)$,
Type 3: $i(D) < \mathfrak{s}(e_D)$,
Type 4: $\mathfrak{s}(e_D) = i(D)$.

Let $\mathcal{F}_\alpha$ be the family of copies $D \in \mathcal{F}$ of Type α for $\alpha \in [4]$. Further, for $i \in [\tau]$ let $\mathcal{F}_\alpha^i = \{D \in \mathcal{F}_\alpha : i(D) = i\}$. Next we find upper bounds on $|\mathcal{F}_\alpha|$ for every $\alpha \in [4]$.

Claim 3.2. $|\mathcal{F}_1| \leqslant n^{v(F)-1}$.

Proof. First observe that for $D \in \mathcal{F}_1$ we must have $e_{j(D)} = e_{i(D)-1}$. Indeed, from the definition of Type 1 and G it follows that in $H_{j(D)} \cup e_{i(D)}$ there is a copy of F containing $e_{i(D)}$. Hence, $e_{i(D)} \in E(H_{j(D)+1})$. Due to the definition of G, this means that $i(D) = j(D) + 1$.

Now fix $j \in [\tau]$ with $j \geqslant 2$ and let $v \in e_{j-1} \smallsetminus e_j$. By the above, every $D \in \mathcal{F}_1^j$ must contain v. In particular, every $D \in \mathcal{F}_1^j$ contains the $k+1$ vertices in the set $e_j \cup v$. Since no $D \in \mathcal{F}$ satisfies $i(D) = 1$, it thus follows that for every $j \in [\tau]$ we have $|\mathcal{F}_1^j| \leqslant n^{v(F)-(k+1)}$. Summing over all $j \in [\tau]$, we obtain

$$|\mathcal{F}_1| = \sum_{j\in[\tau]} |\mathcal{F}_1^j| \leqslant \binom{n}{k} \cdot n^{v(F)-k-1} = n^{v(F)-1} \,,$$

as desired. □

To show the remaining upper bounds, we first observe the following for $\alpha \in \{2, 3, 4\}$. Since for every $D \in \mathcal{F}_\alpha$, it holds that $|e_D \cup e_{i(D)}| \geqslant k + 1$ and $e_D \cup e_{i(D)} \subseteq V(D)$, we have $|\{D \in \mathcal{F}_\alpha^i : e_D = e\}| \leqslant n^{v(F)-|e_D\cup e_{i(D)}|} \leqslant n^{v(F)-k-1}$ for every $e \in V^{(k)}$. Further, note the trivial fact that if for some $e \in V^{(k)}$ there is no $D \in \mathcal{F}_\alpha^i$ with $e_D = e$, then $|\{D \in \mathcal{F}_\alpha^i : e_D = e\}| = 0$. Thus, we infer

$$\begin{aligned} |\mathcal{F}_\alpha| &= \sum_{i \in [\tau]} |\mathcal{F}_\alpha^i| = \sum_{i \in [\tau]} \sum_{e \in V^{(k)}} |\{D \in \mathcal{F}_\alpha^i : e_D = e\}| \\ &\leqslant \sum_{i \in [\tau]} \left|\{e \in V^{(k)} : e = e_D \text{ for some } D \in \mathcal{F}_\alpha^i\}\right| \cdot n^{v(F)-(k+1)} . \end{aligned} \tag{3.2}$$

Next we show that every $e \in V^{(k)}$ can only be counted once in the above sum.

Claim 3.3. *For every $\alpha \in \{2, 3, 4\}$ and distinct $i, j \in [\tau]$ we have*

$$\{e \in V^{(k)} : e = e_D \textit{ for some } D \in \mathcal{F}_\alpha^i\} \cap \{e \in V^{(k)} : e = e_D \textit{ for some } D \in \mathcal{F}_\alpha^j\} = \varnothing .$$

Proof. Observe that for every $D \in \mathcal{F}_2$ we must have $\mathfrak{s}(e_D) = i(D) - 1$. Indeed, by the definition of Type 2 and G, we know that in $H_{\mathfrak{s}(e_D)} \cup e_{i(D)}$ there is a copy of F containing $e_{i(D)}$. Keeping in mind that $\mathfrak{s}(e_D) < i(D)$ and the definition of the F-process, we conclude $i(D) = \mathfrak{s}(e_D)+1$. Therefore, once we fix $i \in [\tau]$, we know that for every $D \in \mathcal{F}_2^i$ we must have $\mathfrak{s}(e_D) = i-1$. This in turn entails that the sets $\{e \in V^{(k)} : e = e_D \text{ for some } D \in \mathcal{F}_2^i\}$ are disjoint for distinct $i \in [\tau]$.

Note that for every $D \in \mathcal{F}_3$ we know that $H_{i(D)} \cup e_D$ contains a copy of F with e_D being an edge of this copy, and that $e_D \notin E(H_{i(D)})$ by the definition of Type 3. Hence, the definition of the F-process implies that $e(D)$ is added in the next step, i.e., $\mathfrak{s}(e_D) = i(D)+1$. Therefore, once we fix $i \in [\tau]$, we know that for every $D \in \mathcal{F}_3^i$ we must have $\mathfrak{s}(e_D) = i + 1$. This in turn entails that the sets $\{e \in V^{(k)} : e = e_D \text{ for some } D \in \mathcal{F}_3^i\}$ are disjoint for distinct $i \in [\tau]$.

For $D \in \mathcal{F}_4$, we have $\mathfrak{s}(e_D) = i(D)$ by definition of Type 4. Therefore, once we fix $i \in [\tau]$, we know that every $D \in \mathcal{F}_4^i$ must satisfy $\mathfrak{s}(e_D) = i$ and the claim follows. □

Claim 3.3 reveals that for $\alpha \in \{2, 3, 4\}$ we have

$$\sum_{i \in [\tau]} \left|\{e \in V^{(k)} : e = e_D \text{ for some } D \in \mathcal{F}_\alpha^i\}\right| \leqslant \binom{n}{k} .$$

Combining this with (3.2) yields

$$|\mathcal{F}_\alpha| \leqslant n^{v(F)-(k+1)} \cdot \binom{n}{k} \leqslant n^{v(F)-1} . \tag{3.3}$$

Comparing the bounds in (3.3) and the bound in Claim 3.2 with (3.1) entails

$$\delta n^{v(F)} \leqslant |\mathcal{F}| \leqslant \sum_{\alpha \in [4]} |\mathcal{F}_\alpha| \leqslant 5 \cdot n^{v(F)-1} .$$

Due to the chosen hierarchy, this is a contradiction. □

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