Codegree conditions for cycle decompositions and Euler tours in 3-uniform hypergraphs

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Abstract

We show that 3-graphs whose codegree is at least $(2/3+o(1))n$ can be decomposed into tight cycles and admit Euler tours, subject to the trivial necessary divisibility conditions. We also provide a construction showing that our bounds are best possible up to the $o(1)$ term. All together, our results answer in the negative some recent questions of Glock, Joos, Kühn and Osthus.

Keywords: Hypergraphs, Euler tours, Cycles, Decompositions

1 Cycle decompositions and Euler tours

1.1 Cycle decompositions

Given a k-uniform hypergraph H, a *decomposition of* H is a collection of subhypergraphs of H such that every edge of H is covered exactly once. When these subhypergraphs are all isomorphic copies of a single hypergraph F we say that it is an F*-decomposition*. Finding decompositions of hypergraphs is one of the oldest problems in combinatorics. For instance, the old and well-known problem of the existence of designs and Steiner systems (solved only recently by Keevash [10] and independently by Glock, Kühn, Lo and Osthus [7]) can be cast as the problem of decomposing a complete hypergraph into smaller complete hypergraphs of a fixed size. Thanks to these two breakthroughs, our knowledge on hypergraph decompositions has increased substantially; but many open questions remain. We refer the reader to the survey of Glock, Kühn and Osthus [8] for an overview. Here, we focus on decompositions of hypergraphs into cycles, a problem which has connections with the old problem of finding Euler tours.

From now on, unless stated otherwise, the hypergraphs considered will be 3-uniform. Here we focus on decompositions of 3-uniform hypergraphs in which the subhypergraphs are all cycles. For $\ell \geqslant 4$, the *tight cycle of length* ℓ is the 3-graph C_ℓ whose vertices are $\{v_1, v_2, \dots, v_\ell\}$ and whose edges are all the triples of the form $\{v_i, v_{i+1}, v_{i+2}\}$ for $1 \leqslant i \leqslant \ell - 2$, together with $\{v_{\ell-1}, v_\ell, v_1\}$ and $\{v_\ell, v_1, v_2\}$. For brevity, and since no other kind of hypergraph cycles will be considered, we will refer to tight cycles as *cycles*.

Given a vertex x in H the *degree of* x, $\deg(x)$, is the number of edges that contain x. Moreover, for a positive integer k, when the degree of every vertex of a hypergraph H is divisible by k we say that H is k*-vertex-divisible*. Note that in a cycle of length $\ell \geqslant 4$, every vertex has degree exactly 3. This implies that, for any $\ell \geqslant 4$, any 3-graph H which admits a C_ℓ-decomposition for $\ell \geqslant 4$ must necessarily be 3-vertex-divisible.

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Another obvious necessary condition to find C_ℓ-decompositions in H is that the total number of edges of H is divisible by ℓ. If H satisfies these two conditions, we say that H is *C_ℓ-divisible*.

However, C_ℓ-divisibility is not enough to guarantee the existence of C_ℓ-decompositions. For instance, a cycle $C_{2\ell}$ is C_ℓ-divisible, but clearly does not have a C_ℓ-decomposition. Perhaps more worryingly, the computational problem of deciding the existence of an F-decomposition in a graph G is known to be NP-complete as soon as a component of F contains more than two edges [5]. Thus, for graphs (let alone hypergraphs), for most F we do not expect the existence of an "easy" characterisation of the hypergraphs admitting F-decompositions. This motivates the search of easily-checkable sufficient conditions which, together with the necessary C_ℓ-divisibility, already force the existence of C_ℓ-decompositions.

In graphs, one natural such setting is to consider minimum degree conditions. For graphs, the analogous necessary divisibility conditions to have a C_ℓ-decomposition are that every vertex must have even degree and the total number of edges must be divisible by ℓ, and we say that graphs satisfying both conditions are *C_ℓ-divisible*. We define $\delta_{C_\ell}(n)$ as the minimum d such that every C_ℓ-divisible graph G on n vertices which satisfies $\delta(G) \geqslant d$ has a C_ℓ-decomposition. Barber, Kühn, Lo, and Osthus [2] introduced the technique of *iterative absorption* to study F-decompositions in graphs. This technique is also crucial to our present work, and will be reviewed in more detail in Section 3. In particular, for cycle decompositions in graphs, their work implies that asymptotic upper bounds for $\delta_{C_\ell}(n)$ can be obtained via $\delta^*_{C_\ell}(n)$, which is the minimum degree which guarantees the existence of 'fractional C_ℓ-decompositions'. More precisely, we now know that $\delta_{C_\ell}(n) \leqslant \delta^*_{C_\ell}(n) + o(n)$.

The famous Nash-Williams conjecture [12] says that $\delta_{C_3}(n) \leqslant 3n/4$. This is still open, with the current best upper bound by Delcourt and Postle [4] being $\delta^*_{C_3}(n) \leqslant (d + o(1))\,n$ where $d \approx 0.827$. Recently, Joos and Kühn [9] showed that $\delta^*_{C_\ell}(n)/n$ tends to $1/2$ whenever ℓ goes to infinity (they also showed results for cycles in k-graphs, which play a rôle in our proof as well). In general, for all cycles of odd length $\ell \geqslant 3$, the best known bounds [1,2,9] are

$$\frac{n}{2} < \frac{\ell}{2(\ell-1)} n \leqslant \delta_{C_\ell}(n) \leqslant \delta^*_{C_\ell}(n) + o(n) \leqslant \left(\frac{1}{2} + C\frac{\log \ell}{\ell}\right) n + o(n).$$

For cycles of even length, the problem was solved exactly by Taylor [14] when the host graph is large enough. She showed that $\delta_{C_4}(n) = 2n/3 - 1$ and $\delta_{C_\ell}(n) = n/2$ for all even $\ell \geqslant 8$. In the remaining case C_6, the asymptotic formula $\delta_{C_6}(n) = (1/2 + o(1))n$ is known [1,2].

In hypergraphs, the minimum degree conditions can be generalised in terms of *codegree*. For k-uniform graphs and a set S of $(k-1)$ vertices, we define the *codegree of S*, $\deg(S)$, as the number of edges of H that contain all of S. We denote the minimum codegree of a hypergraph H over all S by $\delta_{k-1}(H)$. For a 3-graph H, $\delta_2(H)$ is the minimum of $\deg(x, y)$ over all pairs $\{x, y\}$. We define the *C_ℓ-decomposition threshold* $\delta_{C_\ell}(n)$ as the minimum d such that every C_ℓ-divisible hypergraph H on n vertices with $\delta_2(H) \geqslant d$ has a C_ℓ-decomposition.

For a general hypergraph F, it is possible to define the analogous F-decomposition threshold $\delta_F(n)$ (to avoid trivial cases, we are always restricted to host graphs which satisfy the necessary divisibility conditions). From the work of Glock, Kühn, Lo and Osthus [7], we now know that for each F there exists $c > 0$ such that $\delta_F(n) \leqslant (1 - c)n$, for sufficiently large n.

We investigate $\delta_{C_\ell}(n)$ in more detail, for 3-uniform hypergraphs. Our first result gives asymptotically tight bounds on $\delta_{C_\ell}(n)$ for all but finitely many values of ℓ.

Theorem 1.1 *Let $\ell \geqslant 9$ satisfy one of the following: $\ell \geqslant 10^7$, or ℓ is divisible by 3. For every $\varepsilon > 0$, and n sufficiently large, $\delta_{C_\ell}(n) \leqslant (2/3 + \varepsilon)n$.*

Say a k-graph has a *cycle decomposition* if it admits a decomposition into cycles. That is, there are edge-disjoint cycles —not necessarily of the same length— which cover every edge exactly once. This notion is weaker than that of having a C_ℓ-decomposition for a fixed ℓ. It is easy to see that any 3-graph having a cycle decomposition must be 3-vertex-divisible. As a corollary of Theorem 1.1, we obtain an upper bound on the minimum codegree sufficient to force a cycle decomposition.

Corollary 1.2 *Any 3-vertex-divisible hypergraph H with $\delta_2(H) \geqslant (2/3 + o(1))|H|$ has a cycle decomposition.*

Glock, Kühn, and Osthus [8, Conjecture 5.6] conjectured that in fact $\delta_2(H) \geqslant (1/2 + o(1))|H|$ should be enough to force a cycle decomposition. However, it turns out that the value '2/3' in Theorem 1.1 and Corollary 1.2 is best-possible (see Theorem 1.5). We further discuss the optimality of our bounds in Section 1.3.

1.2 Euler tours

Our focus on decompositions into cycles is partly motivated by its close connections with the celebrated problem of finding *Euler tours*. Given a k-graph H, a *tour* is a sequence of non-necessarily distinct vertices $v_1, \ldots, v_\ell$

such that, for every $1 \leqslant i \leqslant \ell$ the k consecutive vertices $\{v_i, v_{i+1}, \dots, v_{i+k-1}\}$ induce an edge (understanding the indices modulo ℓ), and moreover all of these edges are distinct. If a hypergraph H contains a tour that covers each edge exactly once, we call it *Euler tour* and we say that H is *Eulerian*.

A primordial result of graph theory was inspired by the problem of finding a route through the bridges of Königsberg which visits each bridge only once [3]. Euler proved that every Eulerian graph must be 2-vertex-divisible (every vertex has even degree), and stated (later proved by Hierholzer and Wiener) that Eulerian graphs can be characterised as the graphs which are connected in addition to being 2-vertex-divisible.

Analogously, for $k \geqslant 3$, it is an easy observation that every Eulerian k-graph must be k-vertex-divisible. However, the characterisation of Eulerian k-graphs is not as simple as for $k = 2$. For instance, the decision problem of whether a given 3-graph is Eulerian is NP-hard [11]. In fact, until recently, it was not even known if complete k-vertex-divisible k-graphs were Eulerian. It was conjectured by Chung, Diaconis, and Graham [3] that indeed that should be the case, at least for sufficiently large complete k-graphs. This was proven to be true by Glock, Joos, Kühn, and Osthus [6], which deduced this from a more general result which finds Euler tours in k-graphs satisfying certain conditions of quasirandomness (of which the complete graphs are a particular case). From their more general result, the following 'minimum codegree' version can be deduced.

Theorem 1.3 *There is a constant $c > 0$ such that any sufficiently large 3-vertex-divisible hypergraph H with $\delta_2(H) \geqslant (1-c)|H|$ is Eulerian.*

The constant c of Theorem 1.3 is fairly small (by inspecting their proof, we estimate $c \leqslant 2^{-10^{12}}$) and therefore improving the minimum codegree condition becomes a natural problem. Their proof is based fundamentally on a reduction to the problem of finding a cycle decomposition instead of an Euler tour. In a similar fashion, we can use Theorem 1.1 to improve the minimum codegree condition and show that $2n/3$ is essentially enough.

Corollary 1.4 *Any 3-vertex-divisible 3-graph H with $\delta_2(H) \geqslant (2/3 + o(1))|H|$ is Eulerian.*

Both in [6, Conjecture 3] and in [8, Conjecture 5.4] it was conjectured that a minimum codegree condition of $(1/2 + o(1))|H|$ should be enough. However, as it will be presented in the discussion of Section 1.3, Corollary 1.4 is asymptotically optimal (see Theorem 1.5).

1.3 Lower bounds and counterexamples

Our three main results (namely Theorem 1.1, Corollary 1.2 and Corollary 1.4) hold for 3-graphs H satisfying $\delta_2(H) \geqslant (2/3 + o(1))|H|$. It turns out that the '2/3' in those statements cannot be lowered. We prove this by constructing a family of counterexamples which are able to cover all of the previous settings (C_ℓ-decompositions, cycle decompositions, and Euler tours) in an unified way.

A *tour decomposition* of H is a collection of edge-disjoint tours in H which, together, cover all edges of H. Note that a cycle is precisely a tour which does not repeat vertices, thus both C_ℓ-decompositions and cycle decompositions are particular instances of tour decompositions. An Euler tour is a tour decomposition consisting of a single tour. Thus the following result shows that all of Theorem 1.1, Corollary 1.2 and Corollary 1.4 are asymptotically tight.

Theorem 1.5 *Let $\ell \geqslant 1$ and $n \geqslant 3(\ell + 3)$ be divisible by 18. Then there exists a C_ℓ-divisible 3-graph H on n vertices which satisfies $\delta_2(H) \geqslant (2n - 15)/3$, but does not admit a tour decomposition.*

We finish with remarks on this example for the case of Euler tours. Recall that, for graphs, being Eulerian is equivalent to being 2-vertex-divisible and connected. A natural generalization of connectivity for k-graphs with $k \geqslant 3$, is tight connectivity in k-graphs, which we define now. A *walk* between edges a, b in a k-graph H is a sequence of edges $e_1, \dots, e_r$ such that $e_1 = a$, $e_r = b$ and $|e_i \cap e_{i+1}| = k-1$ for each $1 \leqslant i < r$. We say a k-graph H is *tightly connected* if there is a walk between any pair of edges of H. Observe that a tour yields a walk between any pair of edges contained in it, so in particular any Eulerian k-graph must be tightly connected. For $k = 3$, there exist 3-vertex-divisible 3-graphs H with $\delta_2(H) \geqslant (1/2 - o(1))|H|$ which fail to be tightly connected and thus Eulerian [8, Example 5.5]. In contrast, it is known that $\delta_2(H) \geqslant (1/2 + o(1))|H|$ is enough to guarantee tight connectivity, and even the existence of Hamiltonian cycles [13]. Thus our Theorem 1.5 shows that, in contrast with the graph case, tight connectivity is *not* the main bottleneck for the existence of Euler tours. Indeed, there are deeper structural considerations which are present in any tour decomposition (see, e.g. Lemma 2.5), and we exploit such constraints to build our counterexamples.

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[3] When the problem was posed, there were seven bridges in Königsberg, but they were severely damaged during WWII. In present times, the name of the city is Kaliningrad, and only five bridges exist. This new configuration yields a curious partial solution to the original problem, since now an "Euler walk" which transverses every bridge exactly once is possible. Sadly, there is still no Euler tour in the city.

2 Lower bounds

In this section we prove Theorem 1.5. The following 3-graph will be the basis of our construction.

Definition 2.1 Let n be divisible by 18 and write $n = 18k$. Consider the 3-graph H_n on n vertices, whose vertex set is partitioned into three clusters V_0, V_1, V_2 whose sizes are n_0, n_1, n_2 respectively, and are defined by

$$n_0 = 6k, \qquad n_1 = 6k - 2, \qquad \text{and} \qquad n_2 = 6k + 2. \tag{1}$$

Given a vertex $x \in V(H_n)$, the *label* $l(x)$ of x is i if and only if $x \in V_i$. The edge set of H_n is

$$E(H_n) = \{xyz : l(x) + l(y) + l(z) \not\equiv 0 \bmod 3\}.$$

In words, every 3-set is present as an edge in H_n, except for those which are entirely contained in one of the clusters V_i, or have non-empty intersection with all three clusters. In the remainder of this section, n will always be clear from context, and for a cleaner notation we will just write $H = H_n$.

First, it is not difficult to check that the 3-graph H has large minimum codegree.

Lemma 2.2 *Let $n \in 18\mathbb{N}$. Then $\delta_2(H) \geqslant (2n-12)/3$.*

Given n and $i, j, k \in \{0, 1, 2\}$, let H_{ijk} be the edges xyz of H such that $(l(x), l(y), l(z)) = (i, j, k)$. For instance, H_{001} are the edges which have two vertices in V_0 and one vertex in V_1. We note that equations (1) imply that, for $n = 18k$, all n_0, n_1, n_2 are even, and for all $i \in \{0, 1, 2\}$ we have $n_i \equiv i \mod 3$. Using that, straightforward calculations will reveal the following congruences modulo 3.

Lemma 2.3 *Let $n \in 18\mathbb{N}$. Then*

(M1) *for every $x \in V(H)$, $\deg_H(x) \equiv 1 \bmod 3$, and*

(M2) $|H_{112}| \not\equiv |H_{122}| \bmod 3$.

Since H is not quite 3-vertex-divisible, our counterexample will consist actually of a slight alteration obtained by removing a perfect matching. We can select such a matching without using edges from $H_{112} \cup H_{122}$.

Lemma 2.4 *Let $n \in 18\mathbb{N}$. Then there exists a perfect matching $F \subseteq H \setminus (H_{112} \cup H_{122})$.*

The following lemma is crucial for our analysis. It states that *any* tour in H must satisfy constrained intersection sizes with the edges of H_{112} and H_{122}, even after we allow ourselves to add some extra edges in V_0. For a tour W in H, let W_{112} and W_{122} be the edges in $W \cap H_{112}$ and $W \cap H_{122}$, respectively.

Lemma 2.5 *Let $n \in 18\mathbb{N}$. Let S be edge-disjoint with H and $V(S) \subseteq V_0$, and let W be a tour in $H \cup S$. Then $|W_{112}| \equiv |W_{122}| \bmod 3$.*

Proof. Let $W = w_1 w_2 \cdots w_r$, in cyclic order, and let $P = \sigma_1 \cdots \sigma_r$ be a cyclic word over the symbols $\{0, 1, 2\}$, where $\sigma_i = l(w_i)$. Since W is a tour, it does not repeat edges. Thus we have that $|W_{112}|$ is exactly the same as the number of cyclic appearances of the patterns $F_1 = \{112, 121, 211\}$ in P. Similarly, $|W_{122}|$ is exactly counted by the number of cyclic appearances of $F_2 = \{122, 212, 221\}$ in P.

Define $\Phi(P)$ as follows. Scan the cyclic triples of consecutive symbols of P one by one, and if they belong to $F_1 \cup F_2$, we add the sum of the values of their symbols to $\Phi(P)$. Formally (using the Iverson bracket notation),

$$\Phi(P) = \sum_{1 \leqslant i \leqslant r} [\![\sigma_i \sigma_{i+1} \sigma_{i+2} \in F_1 \cup F_2]\!] (\sigma_i + \sigma_{i+1} + \sigma_{i+2}),$$

where the indices are always understood modulo r.

We aim to show that $\Phi(P) \equiv 0 \bmod 3$. If P consists only of 0s, this is clear. If P consists only of symbols in $\{1, 2\}$, by definition of H there cannot be three consecutive 111 or 222. Thus every pattern of P is in $F_1 \cup F_2$ and therefore $\Phi(P)$ sums every symbol three times, thus $\Phi(P) \equiv 0$.

Thus we can assume P contains all three symbols $\{0, 1, 2\}$. We compute $\Phi(P)$ by scanning the *maximal runs* of consecutive symbols of P which only use symbols in $\{1, 2\}$. Let $R_1, R_2, \ldots, R_k$ be those runs. By maximality of the runs, each R_i is surrounded by two 0 symbols, before and after in P. Fix an arbitrary R_i. If $|R_i| < 3$ then it does not contribute anything to $\Phi(P)$, so assume $|R_i| \geqslant 3$. Crucially, note that each R_i cannot begin with two different symbols in $\{1, 2\}$. Indeed, otherwise, together with the immediately preceding 0 symbol it would form a triplet of the form 012 or 021, but these do not appear in P. By the same reason,

R_i cannot end with two different symbols in $\{1, 2\}$. If $|R_i| = 3$ this would imply that R_i is of the form aaa for some $a \in \{1, 2\}$, which is not possible. Thus we can assume $|R_i| \geqslant 4$, and by our analysis it must be of the form aaQ_ibb, where $a, b \in \{1, 2\}$ and Q_i is a (possibly empty) word using only symbols in $\{1, 2\}$. Then the contribution of R_i to the sum in $\Phi(P)$ would be that each symbol in Q_i is added three times, then a three times (once because of its first appearance, and twice because of its second appearance), and similarly b is counted three times. We deduce $\Phi(P) \equiv 0 \bmod 3$ always.

On the other hand, note that, for $j \in \{1, 2\}$, if $\sigma_i\sigma_{i+1}\sigma_{i+2} \in F_j$, then $\sigma_i + \sigma_{i+1} + \sigma_{i+2} \equiv j \bmod 3$. Thus $\Phi(P) \equiv |W_{112}| + 2|W_{122}| \bmod 3$. Since $\Phi(P) \equiv 0 \bmod 3$ we deduce $|W_{112}| \equiv |W_{122}| \bmod 3$, as desired. □

We are now ready to show the proof of Theorem 1.5.

Proof. Consider the 3-graph $H = H_n$ given in Definition 2.1, and $F \subseteq H \setminus (H_{112} \cup H_{122})$ the perfect matching given by Lemma 2.4. Let $\ell' \in \{4, \dots, \ell + 3\}$ be such that $e(H - F) + \ell' \equiv 0 \bmod \ell$. Since $n = 18k \geqslant 3(\ell + 3)$, we have $|V_0| = 6k \geqslant \ell + 3 \geqslant \ell'$. To $H - F$, we add a cycle C of length ℓ', edge-disjoint from $H - F$, which is entirely contained in V_0. We claim $H' = H - F + C$ has all of the desired properties.

We first check H' is C_ℓ-divisible. We start by checking H' is 3-vertex-divisible. Indeed, let $x \in V(H')$ be arbitrary. We have $\deg_H(x) \equiv 1 \bmod 3$ by Lemma 2.3(M1), we have $\deg_F(x) = 1$ since F is a perfect matching, and $\deg_C(x) \equiv 0 \bmod 3$ since C is a cycle on $\ell' \geqslant 4$ vertices. Thus $\deg_{H'}(x) \equiv 1 - 1 + 0 \equiv 0 \bmod 3$ for all $x \in V(H')$, as required. Also, the number of edges of H' is $e(H') = e(H - F) + \ell'$, which was precisely chosen to be divisible by ℓ, so indeed H' is C_ℓ-divisible.

Now we check H' has large codegree. It suffices to show $H - F$ has large codegree. Removing a perfect matching from H decreases the codegree of every pair at most by 1, therefore by Lemma 2.2, we have $\delta_2(H - F) \geqslant \delta_2(H) - 1 \geqslant (2n - 12)/3 - 1 = (2n - 15)/3$.

Now we prove H' does not have a tour decomposition. First notice that, since $F \subseteq H \setminus (H_{112} \cup H_{122})$, the 3-graph H' still has every edge of H_{112} and H_{122}. For a contradiction, suppose that $W^1, \dots, W^r$ are tours forming a decomposition in H'. Since the tours are edge-disjoint and cover all edges of H', we have $\sum_{1 \leqslant i \leqslant r} |W^i_{112}| = |H_{112}|$ and $\sum_{1 \leqslant i \leqslant r} |W^i_{122}| = |H_{122}|$. By Lemma 2.5 (with C playing the rôle of S), we have $|W^i_{112}| \equiv |W^i_{122}| \bmod 3$ holds for each $1 \leqslant i \leqslant r$. We deduce $|H_{112}| \equiv |H_{122}| \bmod 3$, but this contradicts Lemma 2.3(M2). □

3 Sketch of proof of Theorem 1.1

3.1 *Iterative absorption: a summary*

Our proof uses the technique of *iterative absorption* introduced by Barber, Kühn, Lo and Osthus [2]. This method has been used in several decomposition problems for graphs and hypergraphs. We summarise the main steps tailored for our application of finding C_ℓ-decompositions, for a complete and gentle exposition we refer the reader to the expository article by Barber, Glock, Kühn, Lo, Montgomery and Osthus [1].

Let $H = (V, E)$ be a hypergraph on n vertices and with $\delta_2(H) \geqslant (2/3 + \varepsilon)n$. At the beginning of the proof we will define a family of nested subsets of vertices $V(H) = V_0 \supseteq V_1 \supseteq V_2 \supseteq \cdots \supseteq V_l$ each of which keeps the minimum codegree condition (relative to their size). Moreover, these sets linearly decrease their size. Such a family will be called a "vortex". It it easy to see that such a family exists by a random argument. In fact, we can find a vortex such that the size of V_l is bounded by some constant m, which does not depend on n. Afterwards, we remove a small subgraph A called the "absorber". The absorber will have the following crucial property: it will be edge-disjoint from $H[V_l]$, and for any C_ℓ-divisible $R \subseteq H[V_l]$, both A and $A \cup R$ admit a C_ℓ-decomposition. We will describe the construction of A more precisely later. Since the absorber is small, its removal will mildly affect the degree properties overall. Thus we concentrate on finding edge-disjoint copies of C_ℓ in $H' = H - A$.

Now, the strategy to find a full C_ℓ-decomposition in H' goes step by step along the sets of the vortex, through the so called Cover-Down Lemma. We use it to cover the edges of the larger sets of the vortex and keep the remainder always inside the following smaller subsets. More precisely, for $0 \leqslant i < l$, in the i-th step we have already covered all edges in $V(H')$ with edge-disjoint copies of C_ℓ, except for some which are located inside $H'[V_i]$. To go to the next step we need to cover all edges in $H'[V_i] \setminus H'[V_{i+1}]$. For this, we first greedily find an almost C_ℓ-decomposition of $H'[V_i] \setminus H'[V_{i+1}]$ and then, using the minimum degree conditions, we cover all possible remaining edges which are not fully included in V_{i+1}. We need to execute these steps while also preserving the minimum codegree conditions of the leftover hypergraph inside the next set of the vortex. The Cover-Down Lemma can be proven along the same lines as similar lemmatas appearing e.g. in [1,2]. We remark that in this part, and only for $\ell \geqslant 10^7$ not divisible by 3, we make use of recent results by Joos and Kühn [9] on fractional cycle decompositions.

After l steps, we have found edge-disjoint copies of C_ℓ in H', such that all the non-covered edges lie inside V_l. Note that the set of uncovered edges $R \subseteq H[V_l]$ must necessarily form a C_ℓ-divisible subgraph. At this point we will use the properties of the absorber A, to get that $R \cup A$ has a cycle decomposition. This gives the C_ℓ-decomposition of H, as desired.

To construct A, we will construct a different absorber for each possible C_ℓ-remainder, according to the following definition.

Definition 3.1 Given a positive integer ℓ, a hypergraph H, and a subgraph $R \subseteq H$, we say that a subhypergraph $A \subseteq H$ edge-disjoint with $H[V(R)]$ is an *absorber for* R if both A and $A \cup R$ admit C_ℓ-decompositions.

As mentioned before, the set V_l has size at most m, independent of n. Thus, the number of possible leftover hypergraphs $R_1, R_2, \ldots, R_t$ in $H[V_l]$ is also bounded by a function of m, and thus tiny in comparison with n. If, for each $1 \leqslant i \leqslant t$ there exists $A_i \subseteq H$ which is an absorber for R_i, all A_i are edge-disjoint and each A_i has no edges in $H[V_l]$, then the subgraph $A = \bigcup_{i \in [t]} A_i$ is an absorber suitable for our needs. Hence, the following lemma is enough to finish the proof of Theorem 1.2.

Lemma 3.2 *Let $\varepsilon > 0$, ℓ and m positive integers with $\ell > 4$, and n be sufficiently large. Let H be a hypergraph on n vertices with $\delta_2(H) \geqslant (2/3 + \varepsilon)n$. For any C_ℓ-divisible subhypergraph $R \subseteq H$ on at most m vertices there exists an absorber for R on at most $40m^5\ell$ vertices.*

Most of the work is devoted into the proof of Lemma 3.2, which we explain in more detail below.

3.2 Constructing the absorber: tours and trails

Now we describe in more detail our strategy to find absorbers. Consider a C_ℓ-divisible $R \subseteq H$ for which we want to build an absorber. We are halfway done if R happens to admit a tour-decomposition (which, recall, is an edge-disjoint collection of tours which use all the edges of R). Whenever R has this shape, it is possible to find a 'transformer' between R and an edge-disjoint collection of C_ℓ. More precisely, we will find in H, an edge-disjoint collection R' of copies of C_ℓ and a subgraph T, such that R, R', T are edge-disjoint, and both $R \cup T$ and $R' \cup T$ have a C_ℓ-decomposition. In that case, then $R' \cup T$ will be an absorber for R. This construction can be done by adapting the arguments used in the construction of 'absorbers for long cycles' done by Taylor in the graph case [14, Section 5.1]. Hence, our main goal is to 'transform' an arbitrary leftover hypergraph R into a collection of edge-disjoint tours.

For $\ell \geqslant 3$, define a *trail* as a sequence of vertices $u_1 u_2 \cdots u_k$, possibly with repetitions, such that all the edges $\{u_i, u_{i+1}, u_{i+2}\}$ for $i \in \{1, 2, \ldots, k-2\}$ are present and all of them are different. Moreover, given a trail $P = u_1 u_2 \cdots u_{k-1} u_k$ we say that the ordered pairs (u_2, u_1) and (u_{k-1}, u_k) are the *ends* of P. Observe that due the order in which we take these pairs, this definition is independent of the order in which we take the trail.

Given a $R \subseteq H$, suppose it contains a collection of edge-disjoint trails and tours $P_1, P_2, \ldots, P_k$ and $C_1, C_2, \ldots, C_t$ which covers all edges of R exactly once. We call such a collection a *tour-trail decomposition of* R. Since we aim for tour decompositions of R, we would like to find such decompositions where the number of trails is as small as possible, in the following we describe how to modify such decompositions in order to achieve this goal.

Given a tour-trail decomposition $\mathcal{T}$ of R, we define its *residual digraph* $D(\mathcal{T})$ as the (multi)digraph whose arrows correspond to the ends of the trails in $\mathcal{T}$, taken with multiplicities. Crucially, if some residual digraph $D(\mathcal{T})$ contains both arrows (u, v) and (v, u), then it means that we can 'connect' the trails associated with each arrow. By doing this, we will either replace two trails by a single trail, or replace a trail by a tour. In any case, we obtain a tour-trail decomposition of R in which the number of trails has decreased, which leaves us closer to our goal. Our general idea is to start from a leftover R by selecting a tour-trail $\mathcal{T}$, and based on the structure of $D(\mathcal{T})$, replace R by R', in such a way that R' admits a tour-trail decomposition $\mathcal{T}'$ such that $D(\mathcal{T}')$ is 'simpler' than $D(\mathcal{T})$ (we explain this replacement in more detail below). Iterating this, we will get simpler and simpler residual digraphs. Ultimately we will arrive at a leftover R'', with a tour-trail decomposition $\mathcal{T}''$ whose residual digraph $D(\mathcal{T}'')$ is empty: this means that $\mathcal{T}''$ is actually a tour decomposition, and we can finish as discussed before.

As an intermediate step towards the mentioned plan, the following lemma states that, for an arbitrary leftover, we can replace it with another which contains a very particular kind of tour-trail decomposition.

Lemma 3.3 *Let $\varepsilon > 0$, ℓ and m positive integers with $\ell \geqslant 7$, and n be sufficiently large. Let H be a 3-vertex-divisible on n vertices 3-graph with $\delta_2(H) \geqslant (2/3 + \varepsilon)n$. Let $R \subseteq H$ be C_ℓ-divisible on at most m vertices. Then there exists $T \subseteq H$, edge-disjoint with R, on at most $20m^5\ell$ vertices such that*

(i) *T admits a C_ℓ-decomposition,*

(ii) *$T \cup R$ admits a tour-trail decomposition $\mathcal{T}$ whose residual digraph $D(\mathcal{T})$ is a vertex-disjoint union of directed triangles.*

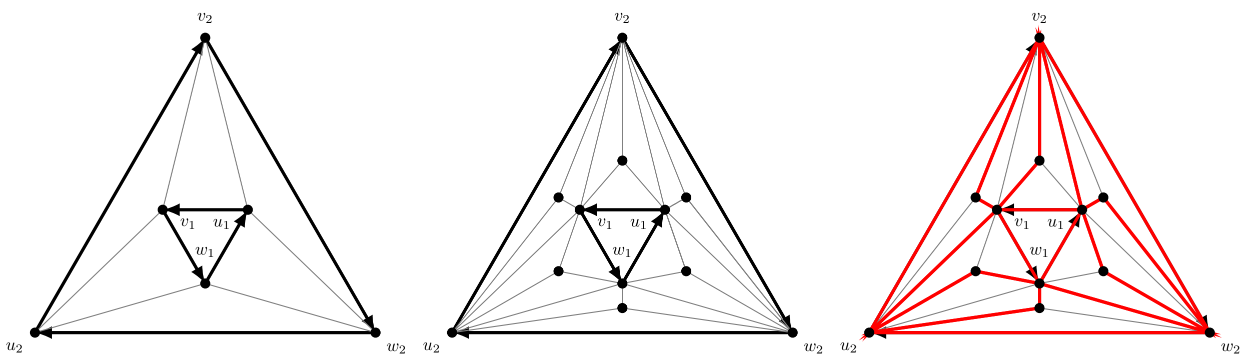
We construct T in Lemma 3.3 as follows. Initially, find an arbitrary tour-trail decomposition of $R_0 = R$ (some must exist, since single edges can form a trail consisting of a single edge). We will find $R_0 \subseteq R_1 \subseteq \cdots \subseteq R_i$, each having a tour-trail decomposition $\mathcal{T}_i$. We will obtain R_{i+1} from R_i by adding a *gadget* G_i to R_i. Such a gadget will consist of a small (at most 20ℓ vertices) C_ℓ-decomposable graph, edge-disjoint with R_i. The point is to identify a small 'repairable structure' in $D(\mathcal{T}_i)$, and the gadget G_i allows us to obtain a new tour-trail decomposition $\mathcal{T}_{i+1}$ in $R_{i+1} = R_i \cup G_i$ where this repairable structure is no longer present in $D(\mathcal{T}_{i+1})$. After a bounded number $q \leqslant m^5$ of steps we will arrive at R_q with a tour-trail decomposition $\mathcal{T}_q$, where $D(\mathcal{T}_q)$ has no repairable structures. In our case this will imply that $\mathcal{T}_q$ has the desired shape (vertex-disjoint union of directed triangles) and the lemma is proven by taking $T = G_1 \cup \cdots \cup G_q$.

Now we describe our gadgets with more detail. We will use large codegree of H to construct the gadgets. The basic observation is the following: given $t \geqslant 5$ and $\varepsilon > 0$, then for n large enough, two pairs (a, b), (c, d) of distinct vertices in $V(H)$, and a small set of vertices $|U| \leqslant \varepsilon n/2$ to avoid, using the minimum codegree $\delta_2(H) \geqslant (2/3 + \varepsilon)n$ we can greedily find a path $P = v_1 v_2 \cdots v_{t-1} v_t$, where $(v_1, v_2, v_{t-1}, v_t) = (a, b, c, d)$ and no vertex of $\{v_3, \ldots, v_{t-2}\}$ is in U. Using this observation repeatedly we can connect pairs to form cycles of controlled length, which then we can decompose into tour-trail decompositions to obtain useful gadgets.

An example of a 'basic gadget' (which will form the basis of more complicated gadgets in our proof) is the following, which we call a *four-cycle gadget*. Suppose R is a given k-graph, and uvw, vwx are two edges not in R. By using the previous observation, we can find an ℓ-cycle C, edge-disjoint from R and containing $uvwx$ as a subpath. Now form a tour-trail decomposition of C by considering the trails $T_1 = uwvx$ and $T_2 = C \setminus \{uvw, vwx\}$ and no tours. It turns out that the residual digraph of this tour-trail decomposition of C is precisely the directed 4-cycle $uvxw$. A similar gadget (with the same residual digraph) can be constructed if the initial two edges are uwx and uvx instead.

As an example of more complicated gadget operations, we will sketch how to use four-cycle gadgets to join trails given by two disjoint directed triangles of a residual digraph. Since disjoint directed triangles in a residual digraph are precisely the output of Lemma 3.3, this will be enough to finish our proof. We remark that the next construction is actually the first place where the full strength of the minimum codegree of at least $(2/3 + \varepsilon)n$ is used, in the previous steps a bound of $(1/2 + \varepsilon)n$ would have sufficed.

Say $C_1 = u_1 v_1 w_1$ and $C_2 = u_2 v_2 w_2$ are the edges of two directed triangles in the residual digraph of an tour-trail decomposition of $R \subseteq H$. We wish to find a C_ℓ-divisible $T \subseteq V(H)$, edge-disjoint with R, such that T admits a tour-trail decomposition whose residual digraph is precisely formed by the two directed triangles $w_1 v_1 u_1$ and $w_2 v_2 u_1$, as they correspond exactly to the opposite directions of C_1, C_2 then they 'cancel out' by joining trails, as described before.



(a) Setting virtual triangles. (b) New common neighbours. (c) Locating cycle gadgets.

Fig. 1. Using $\delta_2(H) \geqslant (2/3 + \varepsilon)n$ and four-cycle gadgets to eliminate two directed triangles from a residual digraph.

To find such T, we follow a geometric intuition. Picture C_1, C_2 as the opposite faces in a triangular prism, where the three square faces are given by $v_1 u_1 v_2 u_2$, $w_1 w_2 u_2 u_1$ and $w_2 w_1 v_1 u_2$. Now decompose each one of the square faces of the prism into two 'virtual' triangles, we say these triangles are 'virtual' because they need not be edges of H. For instance these two triangles in $v_1 u_1 v_2 u_2$ can be $v_1 u_1 v_2$ and $v_2 u_1 u_2$. In the end we obtain a total of six 'virtual' triangles (see Figure 1a). To be able to use four-cycle gadgets we need to find edges in place of the virtual edges, which is done as follows. Using the minimum codegree at least $(2/3+\varepsilon)n$, we can find, for each of the virtual triangles, a common neighbour of its three pairs. For instance, in the virtual triangle $u_1 v_1 v_2$ we find a vertex x which is a common neighbour of the three pairs $u_1 v_1$, $v_1 v_2$ and $u_1 v_2$ (see Figure 1b). In the new structure there are plenty of pairs of edges which intersect in two vertices, and thus suitable to construct four-cycle gadgets using them. By orienting the cycles on those four-cycle gadgets coherently we can get the desired residual digraph (see Figure 1c, where the selected four-cycles are marked in red).

4 Concluding remarks

Let ℓ_0 be the smallest integer such that $\delta_{C_\ell}(n) = (2/3 + o(1))n$ for all $\ell \geqslant \ell_0$. Our main result shows $\ell_0 < 10^7$, and we can give constructions which show that $\ell_0 > 4$. It would be interesting to close this gap.

A natural question is what happens for k-graphs with $k \geqslant 4$. Our proofs and constructions rely on using edges of size 3, and it is not clear to us if Theorem 1.5 indicates the emergence of a pattern where the necessary codegree to ensure cycle decompositions and Euler tours on n-vertex k-graphs is way larger than $(1/2 + o(1))n$.

Question 4.1 *For $k \geqslant 4$, let H be a k-graph on n vertices. Is $\delta_{k-1}(H) \geqslant ((k-1)/k + o(1))n$ a sufficient and asymptotically optimal condition for the existence of cycle decompositions or Euler tours?*

Finally, our proof method naturally suggests a randomised algorithm which outputs an Euler tour in input 3-graphs satisfying $\delta_2(H) \geqslant (2/3+\varepsilon)n$, and runs in expected polynomial time (for fixed ε). We believe standard derandomisation techniques can be applied to transform it into a deterministic polynomial-time algorithm.

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