

Hamiltonicity of inhomogeneous random graphs

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Abstract

We provide a complete characterization of those graphons W for which the inhomogeneous random graph $G(n, W)$ is asymptotically almost surely Hamiltonian. The characterization involves three conditions. Two of them constitute the characterization of $G(n, W)$ being a.a.s. connected, as was shown recently by Hladký and Viswanathan. The third condition captures a geometric obstacle which prevents $G(n, W)$ from having perfect fractional matchings.

1 Introduction

The study of Hamilton cycles, i.e. cycles that traverse every vertex of a graph exactly once, is one of the central themes in graph theory. In the context of random graphs, this inquiry dates back to the very foundations of the field. The classical random models introduced by Gilbert [13] and Erdős and Rényi [8] in 1959, denoted as $G(n, p)$ (where each edge appears independently with probability p), have served as the testbed for understanding the emergence of Hamiltonicity.

For the homogeneous Erdős-Rényi model $G(n, p)$, the emergence of global spanning structures is well understood. The classical results of Erdős and Rényi [8] established that the sharp threshold for connectivity is $p = (1 + o(1)) \frac{\ln n}{n}$. Specifically, for $p = \frac{\ln n + c_n}{n}$, the probability that $G(n, p)$ is connected converges to a limit determined solely by the disappearance of isolated vertices. The problem of Hamiltonicity, while significantly harder, exhibits a striking similarity. A famous result of Pósa [28] determined that the threshold for Hamiltonicity is $p = \Theta(\frac{\ln n}{n})$, using the ‘rotation technique’ that turned out to be very versatile. Independently and at the same time, a much more precise bound on the threshold $p = (1 + o(1)) \frac{\ln n}{n}$ was obtained by Korshunov [23]. This was subsequently refined by Komlós and Szemerédi [22]. Later, a celebrated result by Bollobás [1] unified these observations by showing that in the random graph process, in which edges are added one-by-one and at random starting with the edgeless graph, the obstructions to both properties are purely local: a.a.s. the graph becomes connected the moment the last isolated vertex disappears, and it becomes Hamiltonian exactly when the minimum degree reaches 2. For more details, we

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refer to the relevant chapters in classical monographs (Chapter 6.2 in [12] and Section 8.2 in [2]) on random graphs, as well as a continuously updated survey [11].

However, $\mathbb{G}(n, p)$ assumes homogeneity: every vertex plays the same probabilistic role. This assumption often fails to capture the complexity of real-world networks or more sophisticated mathematical structures, which display intrinsic inhomogeneities such as community structure, or heavy-tailed degree distributions. To study such phenomena, we turn to the theory of graph limits, specifically *graphons*.

A graphon is a symmetric measurable function $W : \Omega^2 \rightarrow [0, 1]$, where (Ω, μ) is an atomless standard probability space with an implicit σ -algebra. Graphons emerged in the work of Borgs, Chayes, Lovász, Sós, Szegedy, and Vesztegombi [25, 4] as the natural limit objects for sequences of dense graphs. Just as every convergent sequence of graphs yields a limit graphon, every graphon W defines a natural model of inhomogeneous random graphs, denoted $\mathbb{G}(n, W)$.

The generation of a random graph $\mathbb{G}(n, W)$ proceeds in two distinct steps, having fixed the vertex set as $V = [n]$:

- (GS1) **Generate vertex types:** We sample n independent points X_1, \dots, X_n from Ω according to the distribution μ . These points are effectively the "types" or "hidden variables" associated with the vertices $1, \dots, n$.
- (GS2) **Generate edges:** Conditioned on the types X_1, \dots, X_n , we form the graph on the vertex set $\{1, \dots, n\}$ by including the edge $\{i, j\}$ with probability $W(X_i, X_j)$, independently for all pairs $1 \leq i < j \leq n$.

When $W(x, y) \equiv p$ is constant, this recovers $\mathbb{G}(n, p)$. However, general W 's allow for rich structural variations. For instance, the Stochastic Block Model corresponds to W being a step function, i.e. there is a partition $\Omega = \bigcup_{i \in [\ell]} S_i$ such that $W|_{S_i \times S_j}$ is constant for all $i, j \in [\ell]$.

In a recent paper [17] the second author and Viswanathan analyzed the connectivity of $\mathbb{G}(n, W)$. The main result of [17] is that $\mathbb{G}(n, W)$ is a.a.s. connected if and only if the following two conditions are satisfied for W :

- (C1) The graphon W is itself *connected*, meaning that whenever Ω is decomposed into two sets $\Omega = S \sqcup T$ with $\mu(S), \mu(T) > 0$ then $\int_{S \times T} W > 0$.
- (C2) The tail of points of small degree in W is light. Specifically, we define the degree of a point $x \in \Omega$ as $\deg_W(x) = \int W(x, y) d\mu(y)$ and for $\alpha \in [0, 1]$ we define $D_W(\alpha) := \{x \in \Omega : \deg_W(x) \leq \alpha\}$. Then we require that $\lim_{\alpha \searrow 0} \frac{\mu(D_W(\alpha))}{\alpha} = 0$.

Condition (C1) is necessary for the absence of macroscopic disconnections, that is, partitions $V = V_1 \sqcup V_2$ with $|V_1|, |V_2| = \Theta(n)$ with no edges between V_1 and V_2 . Condition (C2) is necessary for the absence of isolated vertices.

In this paper, we address the much stronger property of Hamiltonicity in $\mathbb{G}(n, W)$. One might hope that, similar to $\mathbb{G}(n, p)$, the conditions for connectivity (C1) and (C2) might suffice for Hamiltonicity. As it turns out, inhomogeneity introduces a third, purely structural obstacle. More precisely, conditions (C1) and (C2) alone do not guarantee that $\mathbb{G}(n, W)$ has a perfect fractional matching, which is clearly necessary for Hamiltonicity. Using an LP duality perspective, we can distill the obstacle for $\mathbb{G}(n, W)$ to have a perfect fractional matching into a notion we call a 'peninsula'. We motivate and define this notion in the next

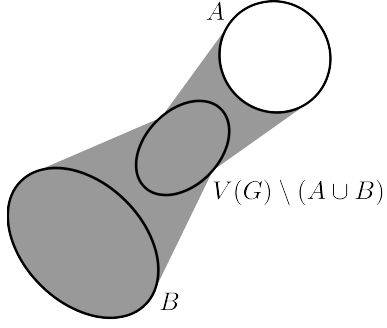


Figure 1: *Left*: A narrow graph peninsula; dark regions indicate admissible edges. *Right*: A beautiful peninsula. Included for pleasure.

section. Our main theorem, Theorem 1.2, then says that the absence of peninsulae plus the connectivity conditions (C1) and (C2) indeed yield that $G(n, W)$ is a.a.s. Hamiltonian.

1.1 Peninsulae

We start by defining graphon peninsulae and then motivate the definition by looking at the more intuitive setting of finite graphs.

Definition 1.1. *Suppose that we have a graphon $W : \Omega^2 \rightarrow [0, 1]$.*

- (i) *W has a peninsula if there exists a number $a \in (0, \frac{1}{2}]$ and disjoint sets $A, B \subset \Omega$ such that $\mu(A) = a$, $\mu(B) = 1 - 2a$, and W is constant-0 on $A \times (A \cup B)$.*
- (ii) *$W : \Omega^2 \rightarrow [0, 1]$ has a narrow peninsula if there exists a number $a \in (0, \frac{1}{2}]$ and disjoint sets $A, B \subset \Omega$ such that $\mu(A) > a$, $\mu(B) = 1 - 2a$, and W is constant-0 on $A \times (A \cup B)$.*

The inspiration for the notions of peninsula and narrow peninsula stems from the matching theory for finite graphs. In this context, given an n -vertex graph G , we analogously say that for $a \in (0, \frac{1}{2}n]$, a pair of disjoint sets $A, B \subset V(G)$ with $|A| \geq a$ and $|B| = n - 2a$ is a *graph peninsula* if G contains no edge with one end-vertex in A and the other in $A \cup B$. Further, a *narrow graph peninsula* amounts to the same condition but $|A| > a$ and $|B| = n - 2a$. See Figure 1 for an example. Let us explain how this notion connects to matching theory. Recall that a function $f : V(G) \rightarrow [0, 1]$ is a *fractional vertex cover* of G if for every edge uv of G we have $f(u) + f(v) \geq 1$. A fractional vertex cover is *half-integral* if $f : V(G) \rightarrow \{0, \frac{1}{2}, 1\}$. For each half-integral vertex cover f , we can take $A := f^{-1}(0)$ and $B := f^{-1}(\frac{1}{2})$. We then have the following correspondence:

- (P1) a graph peninsula corresponds to a half-integral vertex cover f with total weight $\|f\|_1 \leq \frac{n}{2}$ which is not constant- $\frac{1}{2}$,
- (P2) a narrow graph peninsula corresponds to a half-integral vertex cover f with total weight $\|f\|_1 < \frac{n}{2}$.

By the LP duality, a narrow graph peninsula means that G does not have a perfect fractional matching (and hence not a Hamilton cycle). Actually, this is the easy direction of the LP duality and can be seen directly:

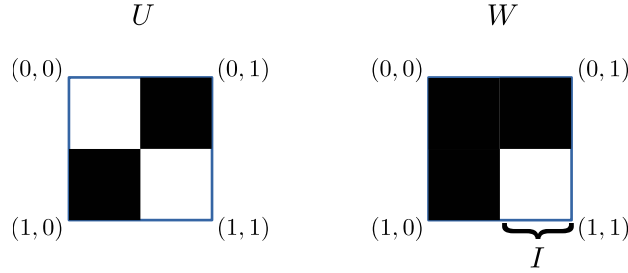


Figure 2: Graphons U and W from Section 1.1

Consider an arbitrary fractional matching. Each edge in a fractional matching with one end in A must have the other end in $V(G) \setminus (A \cup B)$. Hence, the total weight of such a matching on the vertices of A is at most the total weight on $V(G) \setminus (A \cup B)$. As $|A| > |V(G) \setminus (A \cup B)|$, we see that the matching does not cover A , and hence is not perfect.

The meaning of graph peninsulae is more subtle. Indeed, illustrative examples of graphs which do have a graph peninsula are either the complete balanced bipartite graph $H = K_{n/2, n/2}$, or a graph G which is obtained from H by adding edges of $K_{n/2}$ into one of the bipartite classes. In either case, we have a graph peninsula by taking A to be one of the large independent sets and $B = \emptyset$. From this we see that a graph peninsula itself is not an obstruction, as both H and G have a fractional perfect matching and even a Hamilton cycle. However, these fractional perfect matchings/Hamilton cycles are extremely constrained, an issue which will appear fully in the graphon setting. Let U and W be graphons corresponding to H and G (see Figure 2), which both have a peninsula. Due to random fluctuation, $\mathbb{G}(n, U)$ is asymptotically almost surely a complete slightly unbalanced bipartite graph, which does contain a narrow graph peninsula, and thus not a perfect fractional matching. In the random graph $\mathbb{G}(n, W)$, the number of vertices sampled from the constant-0 part (denoted as I in Figure 2) is with probability $\frac{1}{2} + o(1)$ strictly bigger than $\frac{n}{2}$. Since these vertices form an independent set in $\mathbb{G}(n, W)$, they can again serve as a narrow peninsula.

Note that the correspondence (P2) also works in the converse direction. Indeed, if G contains no narrow graph peninsula, then by the nontrivial direction of LP duality, together with the half-integrality of the vertex cover polytope, it follows that G admits a perfect fractional matching. Thus, the absence of narrow peninsulae is equivalent to the existence of a perfect fractional matching. This dual viewpoint indicates that the notions of graph and graphon peninsula provide a natural structural language for formulating if-and-only-if criteria for perfect fractional matchings and, once appropriate additional conditions are imposed, for Hamiltonicity as well.

1.2 Main result

The main theorem is as follows.

Theorem 1.2. *Suppose that $W : \Omega^2 \rightarrow [0, 1]$ is a graphon. Then $\mathbb{G}(n, W)$ is a.a.s. Hamiltonian if the following three conditions are fulfilled:*

- (I) W is a connected graphon,

(II) we have $\lim_{\alpha \searrow 0} \frac{\mu(D_\alpha(W))}{\alpha} = 0$, and

(III) W does not have a peninsula.

Further, these conditions are necessary:

(A) If W is not a connected graphon, then $\lim_n \mathbf{P}[\mathbf{G}(n, W) \text{ is connected}] = 0$.

(B) If $\liminf_{\alpha \searrow 0} \frac{\mu(D_\alpha)}{\alpha} > 0$, then $\liminf_n \mathbf{P}[\mathbf{G}(n, W) \text{ contains an isolated vertex}] > 0$.

(C) If W has a peninsula, then for every $t > 0$, $\limsup_n \mathbf{P}[\mathbf{G}(n, W) \text{ has a matching on at least } n - t \text{ vertices}] \leq \frac{1}{2}$.

Below, we show the easy part of Theorem 1.2, that is, that each of the conditions (A), (B), and (C) is necessary. Then, in Section 1.2.4, we give an outline of the proof.

1.2.1 Necessity of condition (A)

Suppose W is not connected, that is, we have $\Omega = S \sqcup T$ with $\mu(S), \mu(T) > 0$ and $\int_{S \times T} W > 0$. After the vertex-generating step (GS1), asymptotically almost surely, we have that the set $V_S \subset V$ of vertices whose type is in S is nonempty, and that the set $V_T \subset V$ of vertices whose type is in T is nonempty, too. We see that in the edge-generating step (GS2), almost surely, no edge is inserted between V_S and V_T . Hence, $\mathbf{G}(n, W)$ is asymptotically almost surely a disconnected graph.

1.2.2 Necessity of condition (B)

This is shown in Theorem 3(c) in [17]. The idea is that if $\liminf_{\alpha \searrow 0} \frac{\mu(D_\alpha)}{\alpha} > 0$, then after Stage (GS1), asymptotically almost surely, there exists a point X_i with $\deg_W(X_i) \ll \frac{1}{n}$. The corresponding vertex i will, with high probability, end up as an isolated vertex in $\mathbf{G}(n, W)$.

1.2.3 Necessity of condition (C)

Let $A, B \subset \Omega$ be as in Definition 1.1(i). Let $C := \Omega \setminus (A \cup B)$. We have $\mu(C) \leq a$. Let N_A, N_B , and N_C be the random variables counting the number of vertices which in (GS1) get assigned types in A , in B , and in C , respectively. The crucial insight is that in (GS2), almost surely, no edges will be inserted in $\mathbf{G}(n, W)$ between a pair of vertices whose types were both in A , or for which one type was in A and the other in B . This is because W is constant-0 on $A \times (A \cup B)$. This means, that a function $f : [n] \rightarrow [0, 1]$ which assigns to every vertex of type in A value 0, to every vertex of type in B value $\frac{1}{2}$, and to every vertex of type in C value 1 is almost surely a fractional vertex cover of $\mathbf{G}(n, W)$. We have that $\|f\|_1 = \frac{N_B}{2} + N_C = \frac{1}{2}(n + N_C - N_A)$. Thus, if $N_A > N_C + t$, we found a fractional vertex cover of weight less than $\frac{1}{2}(n - t)$, and so there exists no matching (or even fractional matching) of weight at least $\frac{1}{2}(n - t)$. It remains to show that the event $\mathcal{E} = \{N_A > N_C + t\}$ occurs with probability at least $\frac{1}{2} + o(1)$.

The vector (N_A, N_B, N_C) has the multinomial distribution with parameters n and $(a, 1 - 2a, a)$; recall that a is positive but $1 - 2a$ is not necessarily so. The Central Limit Theorem for the multinomial distribution tells us that $\frac{1}{\sqrt{n}}(N_A - an, N_B - (1 - 2a)n, N_C - an)$ converges in distribution to a 3-dimensional centered

normal distribution with covariance matrix Σ ,

$$\Sigma = \begin{bmatrix} a(1-a) & a(1-2a) & a^2 \\ a(1-2a) & 2a(1-2a) & a(1-2a) \\ a^2 & a(1-2a) & a(1-a) \end{bmatrix}.$$

In particular, the event $\mathcal{E} = \{N_A > N_C + t\} = \{\frac{1}{\sqrt{n}}(N_A - an) > \frac{1}{\sqrt{n}}(N_C - an) + \frac{t}{\sqrt{n}}\}$ occurs with probability $\frac{1}{2} + o(1)$, as was needed to show.

1.2.4 Outline of the proof

In Section 2, we recall the necessary background regarding combinatorial optimization (notions of fractional matchings and fractional vertex covers), Szemerédi’s regularity lemma, graphons, probability theory, and functional analysis. The rest of the paper is devoted to the proof of Theorem 1.2. From the assumptions of Theorem 1.2, we establish that for $G = \mathbb{G}(n, W)$ and its regularized cluster graph R (in the sense of the regularity lemma), the following properties hold with high probability:

- (H1) R contains an almost spanning connected component,
- (H2) G has very few vertices of low degree, and
- (H3) R , after minor cleaning, is ‘robustly free of a narrow peninsula’.

When constructing a Hamilton cycle in G using the regularity method, low-degree vertices pose an immediate obstacle. Therefore, in Section 3.1, we construct a small path system \mathcal{P} that covers all such vertices while ensuring its end-vertices lie in high-degree neighborhoods. This allows us to seamlessly append \mathcal{P} to the main bulk of our cycle later in the proof. To construct the bulk of the cycle, we rely on a fractional variant of ‘Łuczak’s connected matchings method’.¹ The goal is to find a near-perfect fractional matching in R , subdivide the regular pairs in its support according to the matching weights, and fill them with almost-spanning paths in G . Connectedness (H1) then allows us to stitch these paths—along with the low-degree path system \mathcal{P} —into a single cycle.

However, a notorious bottleneck in this method arises if the support of the fractional matching is bipartite. In such a setting, even microscopic fluctuations in the cluster sizes or interference from \mathcal{P} make it impossible to perfectly balance the bipartite classes, preventing the paths from forming a fully spanning cycle. It is exactly here that the structural properties from Section 2.2 become vital. In Section 4, we formalize property (H3) via Proposition 4.3, showing that R contains an almost spanning subgraph that is ‘uniquely half-covered’ (equivalently, free of graph peninsulae). As derived in Lemma 2.4, this single property rescues the embedding on two fronts:²

- It guarantees the existence of the almost perfect fractional matching required to build the bulk of the cycle using Łuczak’s connected matchings method.

¹Arguably, the method was first used in [26] and the fractional variant in [5], though in both cases there were almost concurrent and independent developments.

²In fact, in the actual proof, we utilize each of these properties in relation to a different regularization of G , whereas in this simplified overview, we work just with R .

- It guarantees that this subgraph is *not* bipartite.

This guaranteed non-bipartiteness provides the necessary odd cycles to implement the absorption method, systematically introduced by Rödl, Ruciński, and Szemerédi [29] in 2006 and used in many extremal theory results since. In Section 5, we construct an absorber suitable for our purposes. Because we are not trapped in a rigid bipartite parity, this absorber can ‘swallow’ the small, unstructured set of leftover vertices that Łuczak’s connected matchings method could not cover. In Section 6, we put all these components together and construct the desired Hamilton cycle.

2 Preliminaries

2.1 Graphs

In this section we recall two easy graph-theoretic results. Further, deeper graph theory preliminaries concern combinatorial optimization and Szemerédi’s regularity lemma. These are given in separate Sections 2.2, and 2.7, respectively.

We start with the following lemma that provides a walk of odd length³ in any connected non-bipartite graph.

Lemma 2.1. *Let H be a connected non-bipartite graph on r vertices. Then for every pair of vertices $i, j \in V(H)$ (not necessarily distinct), there exists a walk $Q_{i,j}$ of odd length at most $2r - 1$ starting at i and ending at j .*

Proof. Let T be a spanning tree of H . Since H is non-bipartite, there exists an edge $e \in E(H) \setminus E(T)$ that creates an odd cycle C when added to T .

Consider the subgraph $H' = T \cup \{e\}$. In H' , there is a unique path $P_{i,j}$ between i and j utilizing only edges of T , and an alternative walk $W_{i,j}$ that connects i and j by using the odd cycle C via the edge e . Since C has odd length, the length of $P_{i,j}$ and the length of $W_{i,j}$ necessarily have opposite parities. Thus, one of them must be an odd walk.

The length of the tree path $P_{i,j}$ is trivially bounded by $r - 1$. The walk $W_{i,j}$ traverses the edges of $T \setminus E(C)$ at most twice, and the edges of C at most once. Hence its length can also be bounded from above by $2r - 1$. □

By a *binary tree* we mean a tree in which there is exactly one vertex of degree 2 (the *root*), then any number of vertices of degree 3 (*internal vertices*) and any number of vertices of degree 1 (*leaves*). The following claim is easy.

Fact 2.2. *Suppose that T is a binary tree. Then there exists a system \mathcal{F} of paths $\{P_i \subset T\}_i$ with the property that $\{V(P_i)\}_i$ is a partition of $V(T)$ and that the leaves of T are equal to the union of all the leaves of \mathcal{F} .*

2.2 Fractional matchings and fractional vertex covers in graphs

As motivated in Section 1.1, the notion of fractional vertex covers plays a critical role in our proof. While the underlying combinatorial optimization principles we recall below are classical, their application in our set-

³The length of a walk is its number of edges.

ting is highly nontrivial. Specifically, translating the absence of graph peninsulae into the language of fractional vertex covers provides the structural engine for our main argument. It allows us to simultaneously deduce two vastly different macroscopic properties required for our embedding scheme: non-bipartiteness (necessary for constructing absorbing structures) and the existence of fractional perfect matchings (necessary for embedding the bulk of the Hamilton cycle).

We begin by recalling the standard terminology. Suppose that $G = (V, E)$ is a graph. A function $m : E \rightarrow [0, 1]$ is a *fractional matching in G* if for every $v \in V$ we have $\sum_{w \in N_G(v)} m(vw) \leq 1$. The *total weight* of m is defined as $\sum_{e \in E} m(e)$. A fractional matching is *perfect* if its total weight equals $\frac{|V|}{2}$. This is equivalent to having $\sum_{w \in N_G(v)} m(vw) = 1$ for every $v \in V$. The *fractional matching number* of G , denoted $\text{fmn}(G)$, is defined as the supremum of total weights over all fractional matchings in G .

Dually, a function $f : V \rightarrow [0, 1]$ is a *fractional vertex cover in G* if for every $vw \in E$ we have $f(v) + f(w) \geq 1$. The *total weight* of f is defined as $\sum_{v \in V} f(v)$. The *fractional vertex cover number* of G , denoted $\text{fvcn}(G)$, is defined as the infimum of total weights over all fractional vertex covers in G .

The structural properties we rely on stem from three fundamental facts about fractional matchings and fractional vertex covers. The first is arguably the most important instance of LP duality. The second and third facts follow from the half-integrality of the fractional matching polytope (see, e.g., [30, Theorem 30.2]) and the fractional vertex cover polytope (see, e.g., [30, Theorem 64.11]). Recall that a function (whether a fractional matching or a fractional vertex cover) is *half-integral* if its range is a subset of $\{0, \frac{1}{2}, 1\}$.

Fact 2.3. *Suppose that G is a graph.*

- (i) *We have $\text{fmn}(G) = \text{fvcn}(G)$.*
- (ii) *There exists a half-integral matching of total weight $\text{fmn}(G)$.*
- (iii) *There exists a half-integral vertex cover of total weight $\text{fvcn}(G)$.*

We say that a graph G is *uniquely half-covered* if the only half-integral vertex cover of total weight at most $\frac{v(G)}{2}$ is the constant- $\frac{1}{2}$ function. The critical nature of this property becomes obvious when we recall (P1): a graph is a graph peninsula if and only if it is *not* uniquely half-covered.

The following lemma distills the power of this property into the exact two conditions we will need for our main proof. While its proof is a straightforward consequence of the classical theory, this lemma acts as the linchpin of our embedding strategy, guaranteeing the structural integrity of a regularization of $G(n, W)$.

Lemma 2.4. *Suppose that G is a uniquely half-covered graph. Then:*

- (i) *The graph G is not bipartite.*
- (ii) *There exists a half-integral perfect matching in G .*

Proof. We prove part (i) by contrapositive. Suppose that G is bipartite, and let $V(G) = A \sqcup B$ be a bipartition with $|A| \leq |B|$. Then the function $f : V(G) \rightarrow \{0, 1\}$ which is the indicator function of the set A is a half-integral vertex cover of total weight $|A| \leq \frac{v(G)}{2}$. We conclude that G is not uniquely half-covered.

For part (ii), assume that G is uniquely half-covered. Using Fact 2.3(iii), G does not have any fractional vertex cover of weight less than $\frac{v(G)}{2}$. We get that $\text{fvcn}(G) = \frac{v(G)}{2}$. The assertion then follows immediately from Fact 2.3(i) and Fact 2.3(ii). \square

2.3 Graphons

We need some basics of the theory of graphons. We follow the excellent monograph [24]. As mentioned earlier we work with an arbitrary atomless standard measure space Ω with probability measure μ and an implicit σ -algebra. In the theory of graphons, one often works with $\Omega = [0, 1]$ equipped with the Lebesgue measure. This more concrete setting is equivalent to ours by the Lebesgue isomorphism theorem. A *signed graphon* is any symmetric function $W \in L^\infty(\Omega^2)$. Further, we say that W is a *graphon* if $W(x, y) \in [0, 1]$ for μ^2 -almost every $(x, y) \in \Omega^2$. If W is a signed graphon, then its *cut norm* is defined as $\|W\|_{\square} := \sup_{S, T \subset \Omega} \left| \int_{S \times T} W \right|$. In particular, we can introduce the *cut norm distance* between two graphons W_1 and W_2 as $\|W_1 - W_2\|_{\square}$.

A *weighted graph* is a graph $G = (V, E)$ together with a weight function $w : E \rightarrow [0, 1]$. Given a weighted graph (G, w) , we construct a *graphon representation* U as follows. We partition $\Omega = \bigsqcup_{v \in V} \Omega_v$ into sets of measure $\frac{1}{v(G)}$ -each. For a pair $(u, v) \in V^2$, we define $U|_{\Omega_u \times \Omega_v}$ to be constant-0 if uv does not form an edge of G . If it does, we define $U|_{\Omega_u \times \Omega_v}$ to be constant- $w(uv)$. Note that U is not uniquely defined, as it depends on the partition $\Omega = \bigsqcup_{v \in V} \Omega_v$. We say that a *graphon* W is *at the cut distance at most δ from a weighted graph* (G, w) if there exists a graphon representation U as above with $\|W - U\|_{\square} \leq \delta$.

The following lemma can be deduced from well-known results in the theory of graphons. We include a self-contained proof.

Lemma 2.5. *Suppose that $\delta \in [0, 1)$, R is a weighted graph of order n , and W_R is its graphon representation. Suppose that a weighted graph R^* is obtained from R by deleting at most δn vertices. Then R^* has a graphon representation W_{R^*} such that $\|W_R - W_{R^*}\|_{\square} \leq 2\delta$.*

Proof. Suppose that $\Omega = \bigsqcup_{v \in V(R)} \Omega_v$ is a partition of Ω which was used to define W_R . Fix a partition $\Omega = \bigsqcup_{v \in V(R^*)} \Omega_v^*$ such that for every $v \in V(R^*)$ we have $\Omega_v \subset \Omega_v^*$. Take W_{R^*} to be the representation of R^* with respect to this partition. Crucially, observe that for $u, v \in V(R^*)$ and every $(x, y) \in \Omega_u \times \Omega_v$, we have $W_R(x, y) = W_{R^*}(x, y)$. Thus,

$$\|W_R - W_{R^*}\|_{\square} \leq \|W_R - W_{R^*}\|_1 \leq \mu^2 \left(\Omega^2 \setminus \left(\bigcup_{v \in V(R^*)} \Omega_v \right)^2 \right) = 1 - \left(\frac{v(R^*)}{v(R)} \right)^2 \leq 2\delta,$$

as was needed. □

Furthermore, we need two observations about sampling a graph from a graphon. Firstly, the graph degrees are proportional to the graphon degrees. This well-known fact follows from a direct application of the Chernoff bound.

Fact 2.6. *Let W be a graphon and $\varepsilon > 0$. There exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ and for $G \sim \mathbf{G}(n, W)$ the following holds. The probability that for each $i \in [n]$ we have $\deg_G(i) = (\deg_W(X_i) \pm \varepsilon)n$ is at least $1 - \varepsilon$. (Here X_i is the type of the vertex $i \in [n]$ in (GS1).)*

Secondly, a sample is close to the graphon in cut norm distance. This follows from the so-called second sampling lemma [25, Lemma 10.16].

Lemma 2.7. *Let W be a graphon and $\varepsilon > 0$. There exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ and for $G \sim \mathbf{G}(n, W)$ the following holds. The probability that there exists a graphon representation W_G of G such that $\|W_G - W\|_{\square} \leq \varepsilon$ is at least $1 - \varepsilon$.*

We will also make use of the following result which states that a graph which is sufficiently close to a connected graphon cannot have two linear-sized connected components.

Proposition 2.8 (Theorem 1.10(i) in [18]). *Suppose that W is a connected graphon. For every $\zeta > 0$ there exists $\kappa > 0$ with the following property. Suppose that H is a weighted graph whose cut distance from W is at most κ . Then for every set $X \subset V(H)$ with $|X| \in [\zeta v(H), \frac{1}{2}v(H)]$ we have $e_H(X, V(H) \setminus X) > 0$.*

2.4 Probability theory

We need two versions of the Chernoff bound which can for example be derived from [19, Theorem 2.1].

Lemma 2.9. *Suppose that $n \in \mathbb{N}$, $d \in [0, 1]$ and $C \geq 1$. Suppose that B is a random variable with binomial distribution with parameters n and d .*

- (i) $\mathbf{P}[B \leq \frac{1}{2}nd] \leq \exp(-nd/8)$, and
- (ii) $\mathbf{P}[B \geq Cnd] \leq \exp(-ndC \ln(C/e))$.

2.5 Functional analysis

In this section, we collect several definitions and tools from functional analysis tailored to our setting. Throughout, let (Ω, μ) be a probability space with an implicit σ -algebra. For a measurable function $f : \Omega \rightarrow \mathbb{R}$, we define its *support* as $\text{supp}(f) = \{x \in \Omega : f(x) \neq 0\}$. For any $\delta > 0$, we define the δ -*essential support* as $\text{supp}_\delta(f) = \{x \in \Omega : f(x) > \delta\}$.

The space $L^\infty(\Omega)$ is the dual of $L^1(\Omega)$. This duality induces the *weak* topology*. While the underlying machinery of Banach spaces is quite deep, the discrete-minded reader can safely view this topology as a convenient formalism for the “convergence of averages” over arbitrary test sets (for a standard reference, see, e.g., [10, Chapter 5]). For our purposes, we use the following concrete characterization: a sequence of functions $g_1, g_2, \dots \in L^\infty(\Omega)$ *converges weak** to a function $g \in L^\infty(\Omega)$ if, for every measurable set $X \subset \Omega$, we have

$$\lim_{n \rightarrow \infty} \int_X g_n d\mu = \int_X g d\mu.$$

The following fact is trivial to verify. We record it here for a later reference.

Fact 2.10. *Suppose that a sequence functions $g_1, g_2, \dots : \Omega \rightarrow [0, 1]$ converges weak* to a function $g \in L^\infty(\Omega)$. Then we have $g(x) \in [0, 1]$ for almost every $x \in \Omega$.*

A central pillar of our analysis is the *Banach–Alaoglu Theorem*. In the current context, it ensures that the unit ball of $L^\infty(\Omega)$ is weak* sequentially compact. Specifically, given any sequence of functions $g_1, g_2, \dots : \Omega \rightarrow [0, 1]$, there exists a subsequence g_{n_1}, g_{n_2}, \dots and a measurable function $g \in L^\infty(\Omega)$ such that g_{n_k} converges weak* to g as $k \rightarrow \infty$. This compactness-based framework—which allows for the extraction of a continuous limit from a sequence of discrete objects (such as vertex covers of growing finite graphs)—has proven highly effective in the study of graphons, as demonstrated in [16, 7, 14].

2.6 Asymptotic notation

When using multiple parameters, we will sometimes define them via *hierarchies*. Those are defined from right to left. That is we denote by $\beta \ll \alpha$ that β is chosen sufficiently small with respect to α . More precisely, for any $\alpha > 0$ there exists $\beta_0 > 0$ such that for any $0 < \beta \leq \beta_0$ the subsequent statement holds. This extends in the natural way to hierarchies consisting of several parameters.

2.7 Regularity lemma

A substantial part of our proof relies on the regularity method for which we give the necessary background in this section. For two nonempty sets X and Y of vertices of a graph G , the *density* of (X, Y) is defined as $d_G(X, Y) = \frac{e(X, Y)}{|X||Y|}$. A pair (A, B) of disjoint nonempty sets of vertices is ε -regular if $|d_G(A, B) - d_G(A', B')| \leq \varepsilon$ for every $A' \subset A$ and $B' \subset B$ with $|A'| \geq \varepsilon|A|$ and $|B'| \geq \varepsilon|B|$. Subsets of a regular pair inherit some regularity via the following *slicing lemma*.

Fact 2.11 ([21, Fact 1.5]). *Let G be a graph and $\alpha > \varepsilon > 0$. If (X, Y) is an ε -regular pair of density d in G and $X' \subseteq X$ with $|X'| \geq \alpha|X|$ and $Y' \subseteq Y$ with $|Y'| \geq \alpha|Y|$, then (X', Y') is an ε' -regular pair with $\varepsilon' = \max\{\varepsilon/\alpha, 2\varepsilon\}$ and for its density d' we have $|d' - d| < \varepsilon$.*

We say that a partition $V(G) = V_1 \sqcup \dots \sqcup V_\ell$, where the sets V_1, \dots, V_ℓ are called *clusters*, is an ε -regular partition of G if we have

$$(R1) \quad ||V_i| - |V_j|| \leq 1 \text{ for every } i, j \in [\ell], \text{ and}$$

$$(R2) \quad \text{for all but at most } \varepsilon\ell^2 \text{ many pairs of indices } (i, j) \in [\ell]^2 \text{ we have that } (V_i, V_j) \text{ is an } \varepsilon\text{-regular pair.}$$

For constructing the Hamilton cycle our proof is based on the well-known *regularity lemma* in its *degree form*. The theorem below is a simple modification of it, where we start the proof with a prepartition.

Theorem 2.12 ([21, Theorem 1.10]). *For every $\varepsilon > 0$, there exists a constant T such that for every graph G with $V(G) = X \sqcup Y$, where $|X|, |Y| \in \{0\} \cup [\varepsilon n, n]$, and any $d \in [0, 1]$ the following holds. There is an ε -regular partition V_1, \dots, V_t with $\frac{1}{\varepsilon} < t \leq T$ and such that $V_i \subseteq X$ or $V_i \subseteq Y$ for every $i \in [t]$. Moreover, if we let G' be the graph after removing all edges $uv \in E(G)$ with*

- $u, v \in V_i$ for some $i \in [\ell]$,
- $u \in V_i, v \in V_j$, and (V_i, V_j) is not ε -regular, or
- $u \in V_i, v \in V_j$, and (V_i, V_j) has density smaller than d ,

then for every vertex $v \in V(G)$ we have $\deg_G(v) < \deg_{G'}(v) + 2dn$.

For $\varepsilon, d > 0$ and given a graph G with a partition $V(G) = V_1 \sqcup \dots \sqcup V_\ell$ we say that an weighted graph R is the (ε, d) -reduced graph of G with respect to the partition $\{V_i\}_{i \in [\ell]}$ if $V(R) = [\ell]$ and $ij \in E(R)$ if and only if (V_i, V_j) is an ε -regular pair of density at least d . Furthermore, we define the weight of each edge $ij \in E(R)$ by $d_G(V_i, V_j)$. With a suitable representation is the reduced graph close to the original graph in the cut-norm distance.

Fact 2.13. Let G be a graph with an (ε, d) -reduced graph R corresponding to an ε -regular partition of G . Then for every graphon representation W_G of G there exists a graphon representation W_R of R such that

$$\|W_G - W_R\|_{\square} \leq 2\varepsilon + d.$$

The next fact is an easy consequence of the so-called counting lemma (see e.g. [21, Theorem 2.1]). It follows from the fact that every walk of odd length is contained in the blow-up of a shorter walk of odd length.

Fact 2.14. Suppose that $\frac{1}{n} \ll \zeta \ll 1/r \ll \varepsilon \ll d$, and that G is an n -vertex graph with an (ε, d) -reduced graph R on r vertices. If R contains a walk Q of odd length $\ell \leq 2r - 1$ then, for each odd $L \in [\ell, 2r - 1]$, G contains ζn^{L+1} many paths of length L . Further, we may require that the first vertex of each such path is in the cluster represented by the first vertex of Q , and the last vertex is in the cluster represented by the last vertex of Q .

The following fact follows easily from Fact 2.14.

Lemma 2.15. Let $\frac{1}{n} \ll \zeta \ll 1/r \ll \varepsilon \ll d$ and let G be an n -vertex graph with an (ε, d) -reduced graph R with respect to the partition $\{V_i\}_{i \in [r]}$. Suppose that R is connected and non-bipartite. Then there is an odd $L \leq 2r$ such that for $i, j \in V(R)$, not necessarily distinct, and every two sets $X \subseteq V_i$ with $|X| \geq d|V_i|$ and $Y \subseteq V_j$ with $|Y| \geq d|V_j|$, there are at least ζn^{L+1} paths of length L from X to Y .

Proof. For each $i, j \in V(R)$, let $Q_{i,j}$ be a walk of odd length $\ell_{i,j} \leq 2r - 1$ from i to j in R as provided by Lemma 2.1. Let $L := 2r - 1$.

Given vertices $i, j \in V(R)$, not necessarily distinct, and two sets $X \subseteq V_i$ with $|X| \geq d|V_i|$ and $Y \subseteq V_j$ with $|Y| \geq d|V_j|$, consider the collection of clusters $\{V_k\}_{k \neq i,j} \cup \{X\} \cup \{Y\}$, where we ‘replace’ the clusters V_i and V_j with the sets X and Y respectively. Because of Fact 2.11, the $(\varepsilon/d, d/2)$ -reduced graph \hat{R} of $G[\bigcup_{k \neq i,j} V_k \cup X \cup Y]$ is a subgraph of R . In particular, Fact 2.14 applied on \hat{R} and the walk Q_{ij} yields the desired number of paths of length L . \square

We will also make use of the known fact that a regular pair with positive edge-density contains an almost spanning path. This fact follows in a straightforward way from the Blow-up lemma (or by an easy self-contained sequential procedure).

Fact 2.16. Let (A, B) be an (ε, d) -regular pair in a graph G , where $d > 2\varepsilon$ and $|A| = |B|$. Then the bipartite graph $G[A, B]$ contains a path covering all but at most $\varepsilon|A|$ many vertices from A and all but at most $\varepsilon|B|$ many vertices from B .

3 Incorporating low-degree vertices into a path system

Low-degree vertices require special treatment in our construction of a Hamilton cycle, as they cannot be handled directly by the regularity method. Proposition 3.1 produces a path system covering all such vertices in $G(n, W)$; in the proof of Theorem 1.2, this path system will then be incorporated into the bulk of the Hamilton cycle constructed via the regularity method.

Proposition 3.1. Suppose that $W : \Omega^2 \rightarrow [0, 1]$ is a graphon that satisfies the condition of Theorem 1.2(II). Then for every $\varepsilon \in (0, \frac{1}{3})$ there exists $\alpha_0 > 0$ such that for every $\alpha \in (0, \alpha_0/2)$ and for each $n > n_0(\varepsilon, \alpha)$ with probability at least $1 - \varepsilon$ the random graph $G = G(n, W)$ contains a system \mathcal{P} of vertex-disjoint paths with the following properties:

- (a) each path of \mathcal{P} contains at least 3 vertices,
- (b) every vertex of degree less than αn is contained in some path of \mathcal{P} ,
- (c) the endvertices of each path have degree at least αn ,
- (d) we have $|V(\mathcal{P})| < \alpha \varepsilon n$, and
- (e) we have $|\mathcal{P}| < 2/\alpha$.

Proof. Suppose that $\varepsilon \in (0, \frac{1}{3})$ is given. We use the condition of Theorem 1.2(II) to find $\alpha_0 > 0$ so that for every $\beta \in (0, \alpha_0]$ we have

$$\mu(D_W(\beta)) < \varepsilon^2 \beta / 999. \quad (1)$$

Let $\alpha \in (0, \alpha_0/2]$ be fixed. Suppose that n is sufficiently large. Let $j^* \in \mathbb{N}$ be such that

$$2^{-j^*} \alpha_0 \in \left(\frac{100}{\varepsilon n}, \frac{200}{\varepsilon n} \right]. \quad (2)$$

We have $j^* = \Theta(\log n)$.

Throughout much of the proof, we will work with the random types $\{X_i\}_{i \in [n]}$ from (GS1). In the first stage of the proof, we get control on the random degrees of G . In the easy Claim A below, we will relate $\deg_G(i)$ to $\deg_W(X_i)$ up to an additive error $0.01\alpha n$. For those i for which $\deg_W(X_i)$ is small, a more precise control is needed. This is done in Claims B, C, and D. In the second stage of the proof, we use the gained control on the degrees and deterministically built the path system \mathcal{P} .

Claim A. Let \mathcal{D} be the event that for each $i \in [n]$ we have $\deg_G(i) = \deg_W(X_i)n \pm 0.01\alpha n$. Then we have $\mathbf{P}[\mathcal{D}] \geq 1 - \frac{\varepsilon}{3}$.

Proof of Claim A. This follows immediately from Fact 2.6. ◀

Claim B. Let \mathcal{T} be the event that $\{X_i : i \in [n]\} \cap D_W(2^{-j^*} \alpha_0) = \emptyset$. Then we have $\mathbf{P}[\mathcal{T}] \geq 1 - \frac{\varepsilon}{4.8}$.

Proof of Claim B. Indeed,

$$\mathbf{P}[\mathcal{T}] = (1 - \mu(D_W(2^{-j^*} \alpha_0)))^n \stackrel{(1),(2)}{\geq} \left(1 - \frac{200\varepsilon}{999n}\right)^n \geq \exp(-\varepsilon/4.8) \geq 1 - \varepsilon/4.8,$$

as was needed. ◀

Let \mathcal{E} be the event that \mathcal{T} holds and that for every $\beta \in (0, \alpha_0]$ we have $|\{i \in [n] : X_i \in D_W(\beta)\}| < \varepsilon \beta n / 20$.

Claim C. We have $\mathbf{P}[\mathcal{E}] \geq 1 - \frac{\varepsilon}{3}$.

Proof of Claim C. For each $k \in \{0, \dots, j^* - 1\}$, let \mathcal{A}_k be the event that $|\{X_i : i \in [n]\} \cap D_W(2^{-k} \alpha_0)| > 2^{-(k+1)} \varepsilon \alpha_0 n / 20$. Clearly, $\mathbf{P}[\mathcal{E}] \geq \mathbf{P}[\mathcal{T}] - \sum_{k=0}^{j^*-1} \mathbf{P}[\mathcal{A}_k] \geq 1 - \frac{\varepsilon}{4.8} - \sum_{k=0}^{j^*-1} \mathbf{P}[\mathcal{A}_k]$ by Claim B.

Fix $k \in \{0, \dots, j^* - 1\}$. The number of points X_i for which $X_i \in D_W(2^{-k} \alpha_0)$ is binomial with parameters n and $\mu(D_W(2^{-k} \alpha_0)) \stackrel{(1)}{\leq} 2^{-k} \varepsilon^2 \alpha_0 / 999$. The Chernoff bound (Lemma 2.9(ii) with $C = 999/(40\varepsilon)$) gives $\mathbf{P}[\mathcal{A}_k] \leq \exp\left(-\frac{2^{-k} \varepsilon^2 \alpha_0}{999} \cdot \frac{999}{40\varepsilon} \ln\left(\frac{999}{40\varepsilon}\right) \cdot n\right) \leq \exp\left(-\frac{2^{-k} \varepsilon \alpha_0}{40} \cdot \ln\left(\frac{9}{\varepsilon}\right) \cdot n\right)$. For $k = j^* - 1$ note that $\frac{200}{\varepsilon n} \leq 2^{-(j^*-1)} \alpha_0$ by (2), hence the previous bound gives $\mathbf{P}[\mathcal{A}_{j^*-1}] \leq \exp\left(-\frac{5}{n} \cdot \ln\left(\frac{9}{\varepsilon}\right) \cdot n\right) \leq \left(\frac{\varepsilon}{9}\right)^5$. Therefore for every $k \in \{0, \dots, j^* - 1\}$ we have

$$\exp\left(-\frac{2^{-k} \varepsilon \alpha_0}{40} \cdot \ln\left(\frac{9}{\varepsilon}\right) \cdot n\right) \leq \exp\left(-\frac{2^{-(j^*-1)} \varepsilon \alpha_0}{40} \cdot \ln\left(\frac{9}{\varepsilon}\right) \cdot n\right)^{2^{j^*-k-1}} \leq 2^{k-j^*} \cdot \frac{\varepsilon}{9}.$$

Thus $\sum_{k=0}^{j^*-1} \mathbf{P}[\mathcal{A}_k] \leq \sum_{i=1}^{\infty} 2^{-i} \cdot \frac{\varepsilon}{9} \leq \frac{\varepsilon}{8}$ proving the claim. \blacktriangleleft

For $i \in [n]$, let \mathcal{R}_i be the event that $X_i \in D_W(\alpha_0) \setminus D_W(2^{-j^*}\alpha_0)$ and at the same time $\deg_G(i) < \deg_W(X_i)n/2$.

Claim D. We have $\mathbf{P}[\bigcup_{i \in [n]} \mathcal{R}_i] \leq \varepsilon/3$.

Proof of Claim D. Observe that for \mathcal{R}_i to occur, we have to have at least one index $j \in [j^*]$ for which $X_i \in D_W(2^{-(j-1)}\alpha_0) \setminus D_W(2^{-j}\alpha_0)$ and $\deg_G(i) < \deg_W(X_i)n/2$. Observe that conditioning on $X_i = x$ (subject to $x \in D_W(2^{-(j-1)}\alpha_0) \setminus D_W(2^{-j}\alpha_0)$), we have that $\deg_G(i)$ is a binomial random variable with parameters $n-1$ and $\deg_W(X_i)$. Hence, the Chernoff bound (Lemma 2.9(i)) tells us that

$$\mathbf{P}[\deg_G(i) < \deg_W(X_i)n/2 \mid X_i = x] \leq \exp(-2^{-(j-1)}\alpha_0 n/20),$$

where we used that $\deg_W(X_i) \geq 2^{-j}\alpha_0$. Define a function $F : [j^*] \rightarrow (0, 1]$ by $F(j) = \exp(-2^{-(j-1)}\alpha_0 n/20)$. Thus, for any $i \in [n]$, we have

$$\mathbf{P}[\mathcal{R}_i] \leq \sum_{j=1}^{j^*} \mu(D_W(2^{-j}\alpha_0))F(j) \stackrel{(1)}{\leq} \sum_{j=1}^{j^*} \underbrace{\frac{\varepsilon^2 \cdot 2^{-j}\alpha_0}{999}}_{(T1)_j} \cdot \underbrace{F(j)}_{(T2)_j}.$$

Let us look at the terms $(T1)_j$ and $(T2)_j$, in the descending order, that is, as j moves from j^* to 1. For each j in this range, we have $\frac{(T1)_{j-1}}{(T1)_j} = 2$. Further, we have $\frac{(T2)_{j-1}}{(T2)_j} = F(j-1) \leq F(j^* - 1) = \exp(-2^{-j^*}\alpha_0 \cdot 4n/20) \stackrel{(2)}{\leq} \exp(-20/\varepsilon) < \frac{1}{9}$. Thus, the sum $\sum_{j=1}^{j^*} \frac{\varepsilon^2 \cdot 2^{-j}\alpha_0}{999} \cdot F(j)$ is bounded above by a geometric series with initial term

$$\frac{\varepsilon^2 \cdot 2^{-j^*}\alpha_0}{999} \cdot F(j^*) \leq \frac{\varepsilon^2 \cdot 2^{-j^*}\alpha_0}{999} \stackrel{(2)}{\leq} \frac{200\varepsilon}{999n} \leq \frac{2\varepsilon}{9n}$$

and common ratio $\frac{1}{9}$. We conclude that $\mathbf{P}[\mathcal{R}_i] \leq \varepsilon/(4n)$. The union bound gives $\mathbf{P}[\bigcup_{i \in [n]} \mathcal{R}_i] \leq \varepsilon/3$. \blacktriangleleft

We shall show that if $\mathcal{D} \cap \mathcal{E} \setminus \bigcup_{i \in [n]} \mathcal{R}_i$ holds, then we get the assertions of the propositions. Claims A, C, and D tell us that the error probability is sufficiently small. The rest of the proof is hence devoted to the construction of the path system \mathcal{P} .

Let $t := |\{i \in [n] : X_i \in D_W(2\alpha)\}|$. Let us enumerate the elements $\{i \in [n] : X_i \in D_W(2\alpha)\}$ as i_1, i_2, \dots, i_t so that $\deg_W(X_{i_1}), \deg_W(X_{i_2}), \dots, \deg_W(X_{i_t})$ is nondecreasing.

Inductively for $r = 1, \dots, t$, we will fix an arbitrary 2-vertex set $U_r \subset [n]$ with the property that $U_r \subset N_G(i_r) \setminus \left(\bigcup_{s=1}^{r-1} U_s \cup \{i_1, \dots, i_{r-1}\}\right)$. Let us argue that this is possible, say at step r . It suffices to show that

$$\deg_G(i_r) \geq 2 + 3(r-1). \quad (3)$$

We can use the defining property of \mathcal{E} with $\beta := \deg_W(X_{i_r})$. We get that $r < \frac{\varepsilon \deg_W(X_{i_r})n}{20}$. By that fact that event \mathcal{R}_{i_r} does not hold, but \mathcal{T} does hold, we have $\deg_G(i_r) \geq \deg_W(X_{i_r})n/2$. Putting these two bounds together, we have established (3). This allows us to choose U_r .

We have just shown that whenever we have $\mathcal{E} \setminus \bigcup_{i \in [n]} \mathcal{R}_i$, we can create a system of vertex-disjoint binary trees that covers all vertices i_1, \dots, i_t and some additional vertices. Indeed, for each vertex i_r (which is either an internal vertex of a binary tree or the root, depending on whether $i_r \in \bigcup_{s=1}^{r-1} U_s$ or not), the vertices of U_r specify the two descendants of i_r . Every vertex outside $\{i_1, \dots, i_t\}$ is a leaf. We can now turn

the system of binary trees into a system of paths \mathcal{F} using Fact 2.2. Take an arbitrary path in \mathcal{F} . All its vertices i except its ends satisfy $X_i \in D_W(2\alpha)$. At the same time, at least a third of its vertices are internal. Hence,

$$|\mathcal{F}| \leq t \leq \varepsilon \alpha n / 10 \quad \text{and} \quad \frac{1}{3} |V(\mathcal{F})| \leq t \leq \varepsilon \alpha n / 10, \quad (4)$$

where we used the definition of \mathcal{E} to get an upper-bound on t .

The system \mathcal{F} could almost serve as \mathcal{P} from the statement of the proposition; in particular it satisfies (a), (b), and (c). The reason being that every vertex of degree at most αn in G has by Claim A their type in $D_W(2\alpha)$ and is hence included as an internal vertex in \mathcal{F} . Similarly, since the end vertices have their type not in $D_W(2\alpha)$ by Claim A they have degree at least $1.99\alpha n \geq \alpha n$ in G .

It remains to show how to modify it in order to get (d) and (e).

Among all collections \mathcal{P} of path systems satisfying properties (a), (b), and (c) of the proposition, and additionally obeying

$$|V(\mathcal{P})| \leq |V(\mathcal{F})| + |\mathcal{F}| - |\mathcal{P}|, \quad (5)$$

we choose \mathcal{P} that minimizes $|\mathcal{P}|$. Observe that this minimization is over a nonempty solution space, since \mathcal{F} itself satisfies all the required conditions. We claim the remaining properties (d) and (e) are then satisfied as well.

Let us first look at (d). By (5), we have $|V(\mathcal{P})| \leq |V(\mathcal{F})| + |\mathcal{F}|$. The desired bound follows by plugging in (4).

We now look at (e). Let U be a set of vertices defined by taking one endpoint for each path in \mathcal{P} .

Observe for every pair of distinct vertices $u, v \in U$ we have that $N_G(u) \setminus V(\mathcal{P})$ and $N_G(v) \setminus V(\mathcal{P})$ are disjoint. Indeed, otherwise, we could merge two paths from \mathcal{P} (which preserves (a), (b), and (c), as well as (5)), contradicting the minimality of $|\mathcal{P}|$. Then,

$$\begin{aligned} n &\geq \sum_{v \in U}^{\text{disjoint}} |N_G(v) \setminus V(\mathcal{P})| \\ &\stackrel{\text{Prop3.1(c), and Prop3.1(d) together with } \varepsilon < \frac{1}{3}}{\geq} |U| (0.99\alpha n - \alpha n / 3) = |\mathcal{P}| (0.99\alpha n - \alpha n / 3). \end{aligned}$$

This yields $|\mathcal{P}| \leq \frac{2}{\alpha}$ as desired. □

4 Typical structure of the cluster graph of $\mathbb{G}(n, W)$

The main part of the Hamilton cycle in our proof of Theorem 1.2 is constructed using the Szemerédi regularity lemma. In this section, we present the main structural results to this end. Namely, Proposition 4.3 asserts that cluster graphs such as those coming from regularizations of $\mathbb{G}(n, W)$ satisfy a certain robust version of the ‘uniquely half-covered’ property.

4.1 Fractional vertex covers in graphons

As previously noted, matching theory—when applied to the cluster graph—plays a crucial role in our proof of Theorem 1.2. The applicability of this approach relies on specific conditions imposed on the graphon W ,

most notably the condition of Theorem 1.2(III). Consequently, we require an extension of matching theory to the graphon setting. Such a framework was developed in [15, 7].

It turns out that between the two dual aspects of matching theory – the “matching” facet and the “vertex cover” facet – the latter behaves more cleanly for graphons. We shall restrict our review to this facet. Notably, we will not require a full “LP-type duality” between vertex covers and matchings at the level of the graphon W ; instead, this transition is handled at the level of the finite cluster graph R of $G(n, W)$.

Given a graphon $W : \Omega^2 \rightarrow [0, 1]$, we say a function $f : \Omega \rightarrow [0, 1]$ is a *fractional vertex cover* of W if, for almost every $(x, y) \in \Omega^2$, we have $f(x) + f(y) \geq 1$ whenever $W(x, y) > 0$. We say that f is *half-integral* if its range is restricted to $\{0, \frac{1}{2}, 1\}$. The observation made in (P1) regarding the correspondence between graph peninsulae and half-integral vertex covers extends naturally to graphons. We record this for later reference.

Fact 4.1. *A graphon peninsula corresponds to a half-integral vertex cover f with $\|f\|_1 = \frac{1}{2}$ that is not the constant- $\frac{1}{2}$ function.*

The main result of this section shows that if a sequence of graphons converges in the cut norm, any sequence of associated half-integral vertex covers yields a half-integral vertex cover of the limit graphon with favourable properties.

Proposition 4.2. *Suppose W_1, W_2, \dots is a sequence of graphons on Ω converging to a graphon W in the cut norm. Let $f_1, f_2, \dots : \Omega \rightarrow \{0, \frac{1}{2}, 1\}$ be a sequence, such that f_n is a half-integral vertex cover of W_n . Then there exists a half-integral vertex cover $f : \Omega \rightarrow \{0, \frac{1}{2}, 1\}$ of W such that:*

- (a) $\mu(f^{-1}(0)) \geq \liminf_{n \rightarrow \infty} \mu(f_n^{-1}(0))$, and
- (b) $\|f\|_1 \leq \liminf_{n \rightarrow \infty} \|f_n\|_1$.

Proof. The proof follows the spirit of Lemma 3 in [7] and Lemma 2.3 in [14], and uses weak* convergence as we introduced in Section 2.5. We begin with a key claim regarding the interaction of weak* limits and the graphon edge weights.

Claim A. *Let $A_n, B_n \subset \Omega$ be sets such that $\int_{A_n \times B_n} W_n = 0$ for each n . Suppose $\mathbf{1}_{A_n} \xrightarrow{w^*} a$ and $\mathbf{1}_{B_n} \xrightarrow{w^*} b$. Then $\int_{\text{supp}(a) \times \text{supp}(b)} W = 0$.*

Proof of Claim A. It suffices to show $\int_{\text{supp}_\delta(a) \times \text{supp}_\delta(b)} W = 0$ for every $\delta > 0$. Suppose for contradiction that this fails for some $\delta > 0$. Then there exist sets $X \subset \text{supp}_\delta(a)$, $Y \subset \text{supp}_\delta(b)$ and $c > 0$ such that

$$\mu^2(\{(x, y) \in X \times Y : W(x, y) \leq c\}) < \frac{\delta^2 \mu(X) \mu(Y)}{5}. \quad (6)$$

By the definition of supp_δ , we have $\int_X a d\mu \geq \delta \mu(X)$ and $\int_Y b d\mu \geq \delta \mu(Y)$. Due to weak* convergence, for sufficiently large n , we have $\mu(X \cap A_n) \geq \frac{\delta}{2} \mu(X)$ and $\mu(Y \cap B_n) \geq \frac{\delta}{2} \mu(Y)$. Then (6) implies

$$\int_{(X \cap A_n) \times (Y \cap B_n)} W(x, y) \geq \left(\frac{\delta^2}{4} - \frac{\delta^2}{5} \right) c \mu(X) \mu(Y) = \frac{\delta^2 c \mu(X) \mu(Y)}{20}. \quad (7)$$

However, since $W_n \rightarrow W$ in cut norm and $\int_{A_n \times B_n} W_n = 0$,

$$\int_{(X \cap A_n) \times (Y \cap B_n)} W \leq \int_{A_n \times B_n} W = \int_{A_n \times B_n} W_n + o(1) = o(1). \quad (8)$$

The contradiction between (7) and (8) proves the claim. \blacktriangleleft

By the Banach–Alaoglu Theorem, we can pass to a subsequence such that $\lim_n \|f_n\|_1 = \liminf_n \|f_n\|_1$ and the sequences $(\mathbf{1}_{f_n^{-1}(0)})_n$, $(\mathbf{1}_{f_n^{-1}(1/2)})_n$, and $(\mathbf{1}_{f_n^{-1}(1)})_n$ converge weak*. Since f_n is a half-integral vertex cover, we have $W_n(x, y) = 0$ for almost all $(x, y) \in f_n^{-1}(0) \times (f_n^{-1}(0) \cup f_n^{-1}(1/2))$. Applying Claim A to $A_n := f_n^{-1}(0)$ and $B_n := f_n^{-1}(0) \cup f_n^{-1}(1/2)$ and their weak* limits a and b , we define $f : \Omega \rightarrow \{0, \frac{1}{2}, 1\}$ as

$$f(x) = \begin{cases} 0 & x \in \text{supp}(a) \\ \frac{1}{2} & x \in \text{supp}(b) \setminus \text{supp}(a) \\ 1 & x \in \Omega \setminus \text{supp}(b) . \end{cases}$$

By Claim A, f is a half-integral vertex cover of W . Finally, we verify the asserted properties. We have

$$\mu(f^{-1}(0)) = \mu(\text{supp}(a)) = \int_{\text{supp}(a)} 1 d\mu(x) \stackrel{\text{Fact 2.10}}{\geq} \int_{\text{supp}(a)} a(x) d\mu(x) = \int_{\Omega} a(x) d\mu(x) = \lim_{n \rightarrow \infty} \mu(f_n^{-1}(0)), \quad (9)$$

which establishes (a), and is also one of two useful inequalities towards (b). The other inequality is obtained by repeating (9) for $\mu(f^{-1}(0) \cup f^{-1}(1/2))$ in place of $\mu(f^{-1}(0))$. We obtain $\mu(f^{-1}(0) \cup f^{-1}(1/2)) \geq \lim_n (\mu(f_n^{-1}(0)) + \mu(f_n^{-1}(1/2)))$. These two inequalities give

$$\begin{aligned} \|f\|_1 &= 1 - \frac{\mu(f^{-1}(0) \cup f^{-1}(1/2))}{2} - \frac{\mu(f^{-1}(0))}{2} \\ &\leq 1 - \frac{1}{2} \lim_{n \rightarrow \infty} (\mu(f_n^{-1}(0)) + \mu(f_n^{-1}(1/2))) - \frac{1}{2} \lim_{n \rightarrow \infty} \mu(f_n^{-1}(0)) = \lim_{n \rightarrow \infty} \|f_n\|_1, \end{aligned}$$

as was needed for (b). \square

4.2 Fractional matchings in the cluster graph

Recall, that given $\delta > 0$ and a graphon W we write $D_W(\delta) := \{x \in \Omega \mid \deg_W(x) < \delta\}$. Analogously, we write for a weighted graph (G, w) that $D_G(\delta) := \{v \in V(G) \mid \deg_G(v) < \delta v(G)\}$, where for $v \in V(G)$ we set $\deg_G(v) := \sum_{u \in V(G)} w(uv)$. Recall also that in Section 2.3 we introduced the notion of the cut distance between a weighted graph and a graphon.

The next proposition serves as the bridge that transfers conditions (I), (II), and (III) of Theorem 1.2 to structural properties of a regularization R of $G(n, W)$. More precisely, it states that after deleting vertices of low degree from R , the resulting graph is connected and uniquely half-covered. Moreover, these properties persist even after the adversarial deletion of an additional small set of vertices.

Proposition 4.3. *Suppose that $W : \Omega^2 \rightarrow [0, 1]$ is a graphon that satisfies conditions (I), (II), and (III) of Theorem 1.2. Then for every $C > 1$ and $\gamma \in (0, \frac{1}{20C})$ there exist $\varepsilon, \alpha_0 \in (0, 1]$ such that for every weighted graph R of order n whose cut distance from W is less than ε the following holds for $R^* := R - D_R(\alpha_0)$ and any set $A \subseteq V(R^*)$ with $|A| \leq C\gamma\alpha_0 n$:*

- (a) *we have $|V(R) \setminus V(R^*)| = |D_R(\alpha_0)| \leq \gamma\alpha_0 n$,*
- (b) *$R^* - A$ is uniquely half-covered, and*
- (c) *$R^* - A$ is connected.*

In our proof of Theorem 1.2, we apply Proposition 4.3 to two regularizations of $G(n, W)$, each serving a different but complementary role. The first regularization (denoted in Section 6 by R , with error parameter ε_1) is used to produce many odd cycles, which, via Lemma 5.2, yield an absorbing path. Here, the only used consequence of Proposition 4.3(b) is that $R^* - A$ is non-bipartite (Lemma 2.4(i)). The second regularization (denoted by R_2 , with error parameter $\varepsilon_2 \ll \varepsilon_1$) is used to construct an almost perfect fractional matching in R_2 , which is then exploited via a Blow-up-lemma-type argument to embed paths forming the bulk of the Hamilton cycle. In this step, non-bipartiteness is irrelevant, while the existence of a (near-)perfect fractional matching is essential (Lemma 2.4(ii)).

Proof of Proposition 4.3. Throughout, we refer to conditions (I), (II), and (III) of Theorem 1.2 simply as (I), (II), and (III). Also, we shall work with a sequence of weighted graphs R_i ($i = 1, 2, \dots$) and their suitably defined spanning subgraphs R_i^* and R_i' . We view R_i^* and R_i' also as weighted, with the weights inherited from R_i . In particular, the notion of degree and the derived notion of the set $D_G(\delta)$ always refers to this weighted setting.

Suppose that the assertion of the theorem does not hold for W and constants $C > 1$ and $\gamma \in (0, \frac{1}{20C})$. By (II) we may choose $\beta_0 > 0$ such that

$$\mu(D_W(2\delta)) < \gamma\delta/2 \text{ for every } \delta < \beta_0. \quad (10)$$

Let κ be given by Proposition 2.8 applied on W with $\zeta := \beta_0/10$. Choose an arbitrary sequence $\delta_1, \delta_2, \dots > 0$ such that $\lim_{i \rightarrow \infty} \delta_i = 0$. For each $i \in \mathbb{N}$, set

$$\varepsilon_i := \frac{\gamma\delta_i^2}{8}. \quad (11)$$

By assumption, for each $i \in \mathbb{N}$, there exists a weighted graph R_i , say of order n_i , with a graphon representation W_{R_i} such that

$$\|W - W_{R_i}\|_{\square} < \varepsilon_i \quad (12)$$

and such that $R_i^* := R_i - D_{R_i}(\delta_i)$ violates at least one of the properties (a)-(c) where δ_i plays the role of α_0 . Choose $i_0 \in \mathbb{N}$ such that

$$4C\delta_i < \kappa \text{ and } \delta_i < \gamma, \beta_0 \text{ for every } i \geq i_0. \quad (13)$$

We make the following observation.

Claim A. *For every $i \geq i_0$ and $\delta \in [\delta_i/2, \beta_0]$ we have $|D_{R_i}(\delta)| < \gamma\delta n_i$.*

Proof of Claim A. Assume for the sake of contradiction that $|D_{R_i}(\delta)| \geq \gamma\delta n_i$ for some $\delta_i/2 \leq \delta \leq \beta_0$ and $i \geq i_0$. This gives $\mu(D_{W_{R_i}}(\delta)) \geq \gamma\delta$. Set $S := D_{W_{R_i}}(\delta) \setminus D_W(2\delta)$. Combined with (10), we have $\mu(S) \geq \frac{\gamma\delta}{2}$. Also, observe that for every $x \in S$, we have $\int_{y \in \Omega} W(x, y) - W_{R_i}(x, y) d\mu(y) > \delta$. This gives

$$\|W - W_{R_i}\|_{\square} \geq \left| \int_{S \times \Omega} W - W_{R_i} d\mu^2 \right| > \frac{\gamma\delta}{2} \cdot \delta = \frac{\gamma\delta^2}{2} \geq \frac{\gamma\delta_i^2}{8} \stackrel{(11)}{=} \varepsilon_i,$$

contradicting (12). ◀

For every $i \in \mathbb{N}$, let n_i^* be the order of $R_i^* := R_i - D_{R_i}(\delta_i)$. We record an immediate consequence of Claim A.

Claim B. *For every $i \geq i_0$, property (a) holds for R_i^* , with δ_i in place of α_0 .*

Since our initial assumption was that at least one of (a)-(c) fails for R_i and δ_i in place of α_0 , we know that there exists a set $A_i \subseteq V(R_i^*)$ with $|A_i| \leq C\gamma\delta_i n_i$ and such that (b) or (c) does not hold. Set $R'_i := R_i^* - A_i$, $n'_i := v(R'_i)$. Let $W_{R'_i}$ be a graphon representation of R'_i given by Lemma 2.5. We have

$$\|W_{R'_i} - W\|_{\square} < 2(C+1)\delta_i + \varepsilon_i < 3C\delta_i. \quad (14)$$

We first note that R'_i has a guaranteed minimum degree.

Claim C. For every $i \geq i_0$, we have $\text{mindeg}(R'_i) \geq \delta_i n'_i / 2$.

Proof of Claim C. Indeed, take an arbitrary vertex $v \in V(R'_i)$. We have $\deg_{R_i}(v) \geq \delta_i n_i$. Thus,

$$\deg_{R'_i}(v) \geq \deg_{R_i}(v) - |D_{R_i}(\delta_i)| - |A_i| \geq \delta_i n_i - |D_{R_i}(\delta_i)| - C\gamma\delta_i n_i \stackrel{\text{Claim A}}{>} \delta_i n_i - (1+C)\gamma\delta_i n_i \stackrel{(13)}{\geq} \delta_i n'_i / 2.$$

◀

Furthermore, R'_i inherits the property that there are few vertices of small degree.

Claim D. For every $i \geq i_0$ and $\beta \in [0, \beta_0/2)$ we have $|D_{R'_i}(\beta)| \leq \beta n'_i / 10$.

Proof of Claim D. If $\beta < \delta_i/2$, then $D_{R'_i}(\beta) = \emptyset$ by Claim C. So assume that $\beta \geq \delta_i/2$ and observe that

$$|D_{R'_i}(\beta)| \leq |D_{R_i}(\beta + |D_{R_i}(\delta_i)| + |A_i|)| \leq |D_{R_i}(\beta + (1+C)\gamma\delta_i)|. \quad (15)$$

Recalling the assumption $\gamma \in (0, \frac{1}{20C})$ we get the bound $\beta + (1+C)\gamma\delta_i \leq \beta + \frac{\delta_i}{10} \leq 1.2\beta$. In particular, this number is less than β_0 , and hence Claim A applies. We continue with (15),

$$|D_{R'_i}(\beta)| \leq |D_{R_i}(1.2\beta)| \stackrel{\text{Claim A}}{<} 1.2\gamma\beta n_i < \beta n'_i / 10,$$

as was needed. ◀

In Claim B, we have already established property (a). The next claim establishes property (c), too.

Claim E. For every $i \geq i_0$, property (c) holds for $R'_i = R_i^* - A_i$.

Proof of Claim E. Suppose for a contradiction that R'_i is not connected. Let $C_1 \subset V(R'_i)$ be a connected component of R'_i of minimum size. Set $C_2 := V(R'_i) \setminus C_1$, and $\Delta := \max_{v \in C_1} \deg_{R'_i}(v)$. Note that $|C_1| \geq \Delta$ since C_1 is a connected component. We first consider the case that $\Delta \geq \frac{\beta_0}{2} n'_i$. In particular, $|C_1| \geq \frac{\beta_0}{2} n'_i$. By (14) and (13) we have that $\delta_{\square}(W_{R'_i}, W) < \kappa$. Therefore by Proposition 2.8 we have $e_{R'_i}(C_1, C_2) > 0$, contradicting that C_1 is a connected component. Next, consider the case that $\Delta < \frac{\beta_0}{2} n'_i$. Observe that $C_1 \subset D_{R'_i}(\Delta/n'_i)$. Therefore,

$$\Delta \leq |C_1| \leq |D_{R'_i}(\Delta/n'_i)| \stackrel{\text{Claim D}}{\leq} \Delta/10,$$

a contradiction, as $\Delta > 0$ by Claim C. ◀

Since R_i, δ_i and A_i are chosen to violate the assertions of the proposition, but we have shown in Claim B and Claim E that (a) and (c) hold for every $i \geq i_0$, we conclude that (b) does not hold. Hence for every $i \geq i_0$, R'_i has an optimal half-integral vertex cover $c_i : V(R'_i) \rightarrow \{0, \frac{1}{2}, 1\}$ which is not constant- $\frac{1}{2}$. We set

$S_i := c_i^{-1}(0)$, $P_i := c_i^{-1}(1)$ and $\Delta_i := \max_{v \in S_i} \deg_{R'_i}(v)$. Since c_i is a half-integral vertex cover of total weight at most $\frac{n'_i}{2}$, we have $\frac{1}{2}|c_i^{-1}(\frac{1}{2})| + |c_i^{-1}(1)| \leq \frac{n'_i}{2} = \frac{1}{2}|c_i^{-1}(0)| + \frac{1}{2}|c_i^{-1}(\frac{1}{2})| + \frac{1}{2}|c_i^{-1}(1)|$, or equivalently,

$$|S_i| \geq |P_i|. \quad (16)$$

Also, since c_i is not constant- $\frac{1}{2}$, S_i is nonempty. Take a vertex $v_i \in S_i$ whose degree is Δ_i . Since $c_i(v_i) = 0$ and c_i is a fractional vertex cover, we see that each neighbor of v_i is in P_i . In particular,

$$|P_i| \geq \Delta_i. \quad (17)$$

To conclude the proof, we distinguish two exhaustive, though not necessarily disjoint, cases.

Case I. For infinitely many i we have $\Delta_i \geq \beta_0 n'_i / 2$. We write $f_i : \Omega \rightarrow \{0, \frac{1}{2}, 1\}$ for the corresponding half-integral vertex cover of $W_{R'_i}$. We use Proposition 4.2 on the sequence of graphons $W_{R'_i}$ that satisfy Case I, together with their half-integral vertex covers f_i . These graphons converge to W by (14). The assumption of Case I gives us that

$$\mu(f_i^{-1}(0)) = \frac{|S_i|}{n'_i} \stackrel{(16),(17)}{\geq} \frac{\Delta_i}{n'_i} \stackrel{\text{Case I}}{\geq} \beta_0 / 2.$$

In particular, the right-hand side of Proposition 4.2(a) is strictly positive. That is, Proposition 4.2 gives that W has a non-constant half-integral vertex cover of L^1 -norm at most $\frac{1}{2}$, contradicting our key assumption (III) by Fact 4.1.

Case II. There exists at least one $i \geq i_0$ for which we have $\Delta_i < \beta_0 n'_i / 2$. Take such an index i . Observe that $S_i \subset D_{R'_i}(\Delta_i / n'_i)$. From the assumption of Case II, Claim D applies on $D_{R'_i}(\Delta_i / n'_i)$. Hence,

$$\Delta_i \stackrel{(17)}{\leq} |P_i| \stackrel{(16)}{\leq} |S_i| \leq |D_{R'_i}(\Delta_i / n'_i)| \stackrel{\text{Claim D}}{\leq} \Delta_i / 10,$$

a contradiction, as $\Delta_i > 0$ by Claim C. □

5 Absorption

In order to convert the almost spanning cycle—obtained via the regularity lemma and Łuczak’s connected matchings method—into a true Hamilton cycle, we employ the absorption method, originally introduced by Rödl, Ruciński, and Szemerédi [29]. The absorber needed for our purposes is constructed in Lemma 5.2.

We say that a graph G is (ζ, X, U, ℓ) -connected if for every pair of vertices $u, v \in X$ there are at least $\zeta|U|^\ell$ paths from u to v whose ℓ internal vertices lie in U .

Definition 5.1. Given $\gamma > 0$, an n -vertex graph G , and a set $U \subseteq V(G)$, we say that a path $\mathcal{Q} \subseteq G$ is a (U, γ) -absorbing path if for every subset of vertices $B \subseteq U$ of size at most γn , there is a path containing all vertices from $V(\mathcal{Q}) \cup B$, with the same ends as \mathcal{Q} .

Lemma 5.2. Given $\varepsilon, \zeta > 0$ and $\ell, t \in \mathbb{N}$ there is a $\gamma' > 0$ such that for every $\gamma \in (0, \gamma')$, the following holds for every sufficiently large $n \in \mathbb{N}$. Let G be a graph on n vertices and $U \subseteq V(G)$ such that G is (ζ, U, U, t) -connected. Suppose that for every $v \in X$ there are at least $\varepsilon n^{2\ell}$ cycles $C_{2\ell+1}$ containing v . Then G contains a (U, γ^2) -absorbing path of size at most γn .

Proof. Throughout the proof we may assume for $n, \ell, t, c \in \mathbb{N}$ and $\gamma, \varepsilon, \varepsilon_1, \zeta > 0$ the constant hierarchy

$$\frac{1}{n} \ll \gamma \ll \varepsilon_1 \ll \frac{1}{c} \ll \varepsilon, \zeta, \frac{1}{\ell}, \frac{1}{t}. \quad (18)$$

Let $C_{2\ell+1}^*(2)$ be the blow-up of $C_{2\ell+1}$ on an auxiliary vertex set, where every vertex is duplicated once except for one vertex. More precisely, $C_{2\ell+1}^*(2)$ is a graph with vertices in $\{a_0, a_1, b_1, \dots, a_{2\ell}, b_{2\ell}\}$ and all edges of the form $a_i a_{i+1}, b_i b_{i+1}, a_i b_{i+1}, b_i a_{i+1}$ for every $i \in [2\ell - 1]$ plus the edges $a_0 a_1, a_0 b_1, a_0 a_{2\ell}$, and $a_0 b_{2\ell}$.

The following claim follows from a well-known result of Erdős.

Claim A. *Every vertex $v \in U$ is contained in at least $\varepsilon_1 n^{4\ell}$ copies of $C_{2\ell+1}^*(2)$, with v playing the role of a_0 . (For this ε_1 depends on ε and ℓ only.)*

Proof of Claim A. Set $\eta := \varepsilon_1 \binom{c}{2}^{2\ell}$. Take an arbitrary vertex $v \in U$ and construct an auxiliary 2ℓ -uniform hypergraph \mathcal{C} with vertices on $V(G) \setminus \{v\}$ and edges given by

$$E(\mathcal{C}) = \{e \subseteq V(G) \setminus \{v\} : e \cup \{v\} \text{ spans a copy of } C_{2\ell+1}\}.$$

Note that by assumption $|E(\mathcal{C})| \geq \varepsilon n^{2\ell}$. If $K_{c, \dots, c}^{(2\ell)}$ denotes the complete 2ℓ -partite 2ℓ -uniform hypergraph with c vertices in each partite class, a classical result of Erdős [9, Corollary 2] together with $\eta \ll \varepsilon$ implies that

$$\mathcal{C} \text{ contains at least } \eta n^{2c\ell} \text{ many copies of } K_{c, \dots, c}^{(2\ell)}. \quad (19)$$

Each edge of such a copy of $K_{c, \dots, c}^{(2\ell)}$ in \mathcal{C} , together with v , represents a copy of $C_{2\ell+1}$ in G in one of the $(2\ell)!$ possible cyclic orderings (with respect to the vertex partition). Since c is large compared to ℓ , an application of the (partite) Ramsey theorem with $(2\ell)!$ many colours entails that every copy of $K_{c, \dots, c}^{(2\ell)}$ in \mathcal{C} contains a copy of $K_{2, \dots, 2}^{(2\ell)}$ in which all edges together with v represent copies of $C_{2\ell+1}$ in G with the same cyclical ordering. We denote by K^{cyc} a copy of $K_{2, \dots, 2}^{(2\ell)}$ in \mathcal{C} satisfying this property. The argument above entails

$$\text{each copy of } K_{c, \dots, c}^{(2\ell)} \text{ in } \mathcal{C} \text{ contains a copy of } K^{\text{cyc}}. \quad (20)$$

Let κ^{cyc} and κ_c denote the number of copies of K^{cyc} and $K_{c, \dots, c}^{(2\ell)}$ in \mathcal{C} , respectively. Observe that each copy of K^{cyc} can be contained in at most $n^{2\ell(c-2)}$ many copies of $K_{c, \dots, c}^{(2\ell)}$. Moreover, each copy of $K_{c, \dots, c}^{(2\ell)}$ contains at most $\binom{c}{2}^{2\ell}$ many copies of K^{cyc} . Combining those observations and considering (19) and (20), we get

$$\eta \cdot n^{2c\ell} \leq \kappa_c \leq \frac{\kappa^{\text{cyc}} n^{2\ell(c-2)}}{\binom{c}{2}^{2\ell}},$$

which implies $\kappa^{\text{cyc}} \geq \varepsilon_1 n^{4\ell}$. This proves the claim, since each such K^{cyc} yields a copy of $C_{2\ell+1}^*(2)$ in G where v plays the role of a_0 . \blacktriangleleft

Consider a copy of $C_{2\ell+1}^*(2)$ in G . We hence view its vertices $\{a_0, a_1, b_1, \dots, a_{2\ell}, b_{2\ell}\}$ as in G . It is easy to see that the following two vertex orderings form paths:

(V1) $a_1, a_2, b_1, b_2, \dots, a_{2i-1}, a_{2i}, b_{2i-1}, b_{2i}, \dots, a_{2\ell-1}, a_{2\ell}, b_{2\ell-1}, b_{2\ell}$, and

(V2) $a_1, a_0, a_{2\ell}, a_{2\ell-1}, \dots, a_2, b_1, b_2, \dots, b_{2\ell}$

Note that both paths start and end in the same vertices and both span the same set of vertices except for $a_0 = v$, which is only contained in the second path. We say that a graph $C \subseteq G$ on 4ℓ vertices is an *absorber for v* if C together with v span a copy of $C_{2\ell+1}^*(2)$ with v taking the role of a_0 .

For every vertex $v \in U$ define $\mathcal{A}_v = \{C : C \text{ is an absorber for } v\}$. Further, let $\mathcal{A} = \bigcup_{v \in U} \mathcal{A}_v$. We have by Claim A that

$$|\mathcal{A}_v| \geq \varepsilon_1 n^{4\ell} \text{ for every } v \in U. \quad (21)$$

Since by assumption U is nonempty and contains at least $\varepsilon n^{2\ell}$ many paths of length $2\ell - 1$, we have $|U|^{2\ell} \geq \varepsilon n^{2\ell}$ implying $|U| \geq \sqrt[2\ell]{\varepsilon} n$. Therefore for $\varepsilon_2 := \varepsilon_1 \sqrt[2\ell]{\varepsilon}$ we get

$$n^{4\ell+1} \geq |\mathcal{A}| \geq |U| \cdot \varepsilon_1 n^{4\ell} \geq \varepsilon_2 n^{4\ell+1}. \quad (22)$$

Claim B. *There exists a set $\hat{\mathcal{A}} \subset \mathcal{A}$ such that*

- (a) $|V(\hat{\mathcal{A}})| \leq \gamma n$,
- (b) $V(C_1) \cap V(C_2) = \emptyset$ for every pair $C_1, C_2 \in \hat{\mathcal{A}}$ with $C_1 \neq C_2$, and
- (c) for every vertex $v \in U$ we have $|\hat{\mathcal{A}} \cap \mathcal{A}_v| \geq \gamma^2 n$.

Proof of Claim B. Since $\gamma \ll \varepsilon_1, \varepsilon_2$ we may chose a constant $\alpha \in (0, 1]$ such that

$$\frac{4\gamma^2}{\varepsilon_1} \leq \alpha \leq \frac{\varepsilon_1 \varepsilon_2 \gamma^{3/2}}{5\ell}. \quad (23)$$

For any $\mathcal{A}_* \subset \mathcal{A}$, let $\mathcal{B}(\mathcal{A}_*) = \{(C_1, C_2) \in \mathcal{A}_*^2 : C_1 \neq C_2 \text{ and } V(C_1) \cap V(C_2) \neq \emptyset\}$. We want to get an upper-bound on $|\mathcal{B}(\mathcal{A})|$. Let us count those pairs $(C_1, C_2) \in \mathcal{A}^2$ for which $C_1 \neq C_2$ and $V(C_1) \cap V(C_2) \ni w$ for a given vertex $w \in V(G)$. In such a situation, C_1 is determined by its remaining 4ℓ vertices and so is C_2 . Summing over all $w \in V(G)$, we get

$$|\mathcal{B}(\mathcal{A})| \leq n \cdot n^{4\ell} \cdot n^{4\ell} = n^{8\ell+1}. \quad (24)$$

Let $\hat{\mathcal{A}}_0$ be a random collection picked from \mathcal{A} , by including each element independently with probability $p = \alpha n / |\mathcal{A}|$. We have $\mathbf{E}[|V(\hat{\mathcal{A}}_0)|] = 2\ell \cdot |\mathcal{A}| \cdot p = 2\ell \cdot \alpha n \leq \gamma n / 2$. By Markov's inequality,

$$\mathbf{P}[|V(\hat{\mathcal{A}}_0)| \geq \gamma n] \leq 1/2. \quad (25)$$

Next, we look at $\mathbf{E}[|\mathcal{B}(\hat{\mathcal{A}}_0)|]$. We have

$$\mathbf{E}[|\mathcal{B}(\hat{\mathcal{A}}_0)|] = p^2 \cdot |\mathcal{B}(\mathcal{A})| \stackrel{(22), (24)}{\leq} \left(\frac{\alpha n}{\varepsilon_2 n^{4\ell+1}} \right)^2 \cdot n^{8\ell+1} \stackrel{(23)}{\leq} \gamma^3 n.$$

By Markov inequality's,

$$\mathbf{P}[|\mathcal{B}(\hat{\mathcal{A}}_0)| \geq 4\gamma^3 n] \leq 1/4. \quad (26)$$

Last, we look at $\mathbf{E}[|\hat{\mathcal{A}}_0 \cap \mathcal{A}_v|]$ for any vertex $v \in U$. We have $\mathbf{E}[|\hat{\mathcal{A}}_0 \cap \mathcal{A}_v|] = p \cdot |\mathcal{A}_v| = \alpha n \cdot \frac{|\mathcal{A}_v|}{|\mathcal{A}|}$. We can now use (22), (21), and (23) and get $\mathbf{E}[|\hat{\mathcal{A}}_0 \cap \mathcal{A}_v|] \geq 4\gamma^2 n$. Therefore, by the Chernoff inequality (Lemma 2.9(i)) and the union bound we obtain

$$\mathbf{P}[\text{there is a vertex } v \in U \text{ with } |\hat{\mathcal{A}}_0 \cap \mathcal{A}_v| \leq 2\gamma^2 n] \leq n \exp(-\gamma^2 n / 2) < 1/4. \quad (27)$$

Thus, with positive probability the random set $\hat{\mathcal{A}}_0$ avoids the properties listed in (25), (26), (27). Let us fix such a set $\hat{\mathcal{A}}_0$.

Finally, we obtain $\hat{\mathcal{A}}$ from \mathcal{A}_0 by removing all absorbers that are contained in some pair in $\mathcal{B}(\hat{\mathcal{A}}_0)$. Since \mathcal{A}_0 avoids the property listed in (25), we have (a). The last removal step ensures (b). Finally, to verify (c), we combine the lower bound $|\hat{\mathcal{A}}_0 \cap \mathcal{A}_v| > 2\gamma^2 n$ from (27) with the fact that the number of removed absorbers in the last step was less than $4\gamma^3 n$ by (26). ◀

Observe that each $C \in \mathcal{A}_v$ can be viewed as a path like in (V1), i.e. not containing v . In other words, $\hat{\mathcal{A}}$ can be viewed a disjoint collection of paths. Moreover, given a set of vertices $U' \subseteq U$ of size at most $\gamma^2 n$, we can greedily assign each vertex $v \in U'$ to its own absorber $C \in \hat{\mathcal{A}} \cap \mathcal{A}_v$. In this way, using the paths from (V2), the family $\hat{\mathcal{A}}$ together with the vertices in U' can also be viewed as a collection of paths, with the same endings as the paths in $\hat{\mathcal{A}}$ (now containing all vertices from U'). This is precisely the absorber property required in Definition 5.1.

We only need to connect the paths in $\hat{\mathcal{A}}$. We do this directly by fixing an ordering $\hat{\mathcal{A}} = \{C_1, \dots, C_r\}$ and iteratively connecting each C_i with the next C_{i+1} . In each case we have at least $\zeta|U|^t \geq \zeta(\sqrt[\ell]{\varepsilon n})^t$ potential connections with t internal vertices from U . At most $r \cdot t \leq \frac{\gamma^t}{2^\ell} n$ many vertices are blocked from previous connections and at most $|V(\hat{\mathcal{A}})| \leq \gamma n$ by vertices of $\hat{\mathcal{A}}$. Hence at any step at most $(\frac{\gamma^t}{2^\ell} + \gamma)n^t = \gamma^{\frac{t+2\ell}{2\ell}} n^t$ paths are blocked for the next connection. Thus, by (18) we always have a straightforward choice to join all paths into the required absorbing path \mathcal{Q} . ◻

6 Proof of the main part of Theorem 1.2

Suppose that W satisfies conditions (I)-(III), and n is sufficiently large. For the rest of the proof, we introduce additional constants with the following hierarchy:

$$0 < \frac{1}{n} \ll \frac{1}{r_2} \ll \varepsilon_2 \ll d_2 \ll \rho \ll \gamma \ll \zeta \ll \bar{\zeta} \ll \frac{1}{r_1} \ll \varepsilon_1 \ll d_1 \ll \alpha \ll \beta \ll 1. \quad (28)$$

Observe that by Proposition 3.1 (applied with 2α in place of α and with $\varepsilon < \alpha/2$) and by Lemma 2.7 we have that $G \sim \mathbb{G}(n, W)$ a.a.s. satisfies all of the following properties.

(F1) G contains a collection of vertex-disjoint paths \mathcal{P} such that

- (i) $D_G(2\alpha) \subseteq V(\mathcal{P})$,
- (ii) the end vertices of each path have degree at least $2\alpha n$,
- (iii) $|V(\mathcal{P})| < \alpha^2 n$, and
- (iv) $|\mathcal{P}| < 2/\alpha$.

(F2) There exists a graphon representation W_G of G with $\|W_G - W\|_{\square} \leq \varepsilon_2/2$.

We proceed to show that then $G = (V, E)$ contains a Hamilton cycle. This will prove the theorem.

Step 1: Covering all low degree vertices with few paths

We fix \mathcal{P} given by (F1) and note that in particular \mathcal{P} contains all vertices of degree less than αn . Set $X := \{v \in V : \deg_G(v) \geq \alpha n\}$ and note that then $V = X \cup V(\mathcal{P})$.

Step 2: Finding an absorber

In this step, we want to use Lemma 5.2 to find an $(X \setminus V(\mathcal{P}), \gamma^2)$ -absorbing path \mathcal{Q} . Recall that by (28) we have $1/r_1 \ll \varepsilon_1$. Hence we may apply Theorem 2.12, this time with a trivial prepartition $V = \emptyset \sqcup V$, and obtain an ε_1 -regular partition V_1, \dots, V_{r_1} of G . To use Lemma 5.2, we need to show that G is $(\zeta, X \setminus V(\mathcal{P}), L+1)$ -connected and that every vertex in X is contained in at least ζn^{L+1} cycles in $X \setminus V(\mathcal{P})$ of length $L+2$, for some odd $L \leq 2r_1$. Let R be the (ε_1, d_1) -reduced graph of G with respect to the partition $\{V_i\}_{i \in [r_1]}$. By (F2) we have $\|W_G - W\|_{\square} \leq \varepsilon_2/2$. By Fact 2.13 there exists a graphon representation W_R such that $\|W_R - W_G\|_{\square} < d_1 + 2\varepsilon_1$. Altogether this gives $\|W - W_R\|_{\square} \leq \|W_R - W_G\|_{\square} + \|W_G - W\|_{\square} \leq \varepsilon_2/2 + d_1 + 2\varepsilon_1 \leq 2d_1$. With our choice of constants, Proposition 4.3 applied with β instead of γ and with $C = \frac{1}{100\beta}$ entails that

$$|D_R(\alpha)| \leq \alpha\beta r_1 \quad (29)$$

and that the subgraph $R^* = R - D_R(\alpha)$ satisfies (b) and (c) for every $A \subseteq V(R^*)$ with $|A| \leq \alpha r_1/100$. In the claim below, we show that this condition holds for a particular choice of A .

Claim C. *Set $A := \{i \in V(R^*) : |V_i \cap V(\mathcal{P})| \geq \beta|V_i|\}$. Then the set A satisfies properties (b) and (c) from Proposition 4.3. Moreover, for every vertex $v \in X$ there exists an index $i_v \in V(R^*) \setminus A$ such that*

$$|(N_G(v) \cap V_{i_v}) \setminus V(\mathcal{P})| \geq \frac{\alpha}{2}|V_{i_v}|.$$

Proof of Claim C. Observe that we have $|A| \leq \frac{|V(\mathcal{P})|}{\beta n} r_1 \stackrel{(F1)(iii)}{\leq} \frac{\alpha^2 r_1}{\beta} < \frac{\alpha r_1}{100}$. Therefore, Proposition 4.3 applies. For the moreover part of the claim, let $v \in X$ and note that by also using (F1)(iii) and (29) we have

$$\begin{aligned} \left| \left(N_G(v) \cap \bigcup_{i \in V(R^*) \setminus A} V_i \right) \setminus V(\mathcal{P}) \right| &\geq \deg_G(v) - |V(\mathcal{P})| - \frac{n}{r_1} (|D_R(\alpha)| + |A|) \\ &\geq \alpha n - \alpha^2 n - \left(\alpha\beta + \frac{\alpha^2}{\beta} \right) n \stackrel{(28)}{\geq} \frac{\alpha n}{2}. \end{aligned}$$

Thus, using an average argument, there is an index $i_v \in V(R^*) \setminus A$ satisfying the conclusion of the claim. \blacktriangleleft

For every $i \in V(R^*) \setminus A$ set $V'_i := V_i \setminus V(\mathcal{P})$ and $V' := \bigcup_{i \in V(R^*) \setminus A} V'_i$. Notice that by Fact 2.11 we have that $\{V'_i\}_{i \in V(R^*) \setminus A}$ is an $(\varepsilon_1/(1-\beta)) \leq 2\varepsilon_1$ -regular partition of $G[V']$ (where the classes might have different sizes up to a factor of $1 \pm \beta$). Let \tilde{R} be the reduced graph defined by $V(\tilde{R}) := V(R^*) \setminus A$ and with edges given by pairs $i, j \in V(\tilde{R})$ where (V'_i, V'_j) forms an $2\varepsilon_1$ -regular pair with density at least $d_1/2$. By Fact 2.11 we have that if a pair (V_i, V_j) is ε_1 -regular with density at least d_1 , then (V'_i, V'_j) is $2\varepsilon_1$ -regular with density at least $d_1/2$. In other words, $R^* - A$ is a subgraph of \tilde{R} .

We are now ready to show the properties needed for applying Lemma 5.2. Due to (c), the graph $R^* - A$ is connected. Further, due to (b) and Lemma 2.4(i), the graph $R^* - A$ is not bipartite. Therefore, Lemma 2.15 yields an odd integer $L \leq 2r_1$ for which there are ζn^{L+1} paths of length L in $G[V']$ between every two sets of vertices $Y \subseteq V'_i$ with $|Y| \geq d_1|V'_i|/4$ and $Z \subseteq V'_j$ with $|Z| \geq d_1|V'_j|/4$. Thus, Claim C entails that for any two vertices $u, v \in X$, there are that many paths of length L between $N_G(v) \cap V'_{i_v}$ and $N_G(u) \cap V'_{i_u}$. Hence,

$$G \text{ is } (\zeta, X, X \setminus V(\mathcal{P}), L+1)\text{-connected,} \quad (30)$$

which yields that G is $(\zeta, X \setminus V(\mathcal{P}), X \setminus V(\mathcal{P}), L + 1)$ -connected as well. Similarly, for every $v \in X$ there are at least ζn^{L+1} paths of length L in $G[V']$, ending and starting in $N_G(v) \cap V'_{iv}$. Hence, this yields as many cycles of length $L + 2$ in $X \setminus V(\mathcal{P})$, containing v .

Putting all this together we can apply Lemma 5.2 (with $X \setminus V(\mathcal{P})$ in place of U) to find an $(X \setminus V(\mathcal{P}), \gamma^2)$ -absorbing path \mathcal{Q} with

$$|V(\mathcal{Q})| \leq \gamma n. \quad (31)$$

Step 3: Choosing a reservoir for connections

We begin with a claim that provides many connections between any two vertices in X .

Claim D. *For every pair of distinct vertices $u, v \in X$, there are at least $4\rho n$ internally disjoint paths from u to v of length $L + 1$ whose internal vertices are in $X \setminus (V(\mathcal{P}) \cup V(\mathcal{Q}))$.*

Proof of Claim D. Recall that $L \leq 2r_1$ and therefore $\rho \ll \gamma \ll \zeta \ll 1/L$ by (28). Hence we get from (30) and (31) that

$$G \text{ is } (\zeta/2, X, X \setminus (V(\mathcal{P}) \cup V(\mathcal{Q})), L + 1)\text{-connected.} \quad (32)$$

Now, given vertices $u, v \in X$, let \mathcal{B} be the largest family of internally disjoint paths of length $L + 2$ from u to v whose internal vertices are in $X \setminus (V(\mathcal{P}) \cup V(\mathcal{Q}))$. Suppose for a contradiction that $|\mathcal{B}| < 4\rho n$. Since \mathcal{B} is maximal, all other paths between u and v contain at least one vertex in $V(\mathcal{B})$. Since $|V(\mathcal{B})| \leq 4(L + 3)\rho n$ the maximum number of paths connecting u and v and containing a vertex in $V(\mathcal{B})$ is at most $4(L + 3)^2 \rho n^{L+1} < \zeta n^{L+1}/2$. This is a contradiction to (32). ◀

The next claim provides a small reservoir set through which every pair of vertices from X can be connected sufficiently many times.

Claim E. *There exists a set $S \subseteq X \setminus (V(\mathcal{P}) \cup V(\mathcal{Q}))$ such that*

- (a) $|S| \leq 2\rho n$ and
- (b) *for every pair of distinct vertices $u, v \in X$, there are at least $\rho^{L+2}n$ internally disjoint paths from u to v in G whose internal vertices are contained in S .*

Proof of Claim E. The proof is using the probabilistic method. Take a random subset of vertices $S \subseteq X \setminus (V(\mathcal{P}) \cup V(\mathcal{Q}))$ by independently including each vertex with probability ρ . By Markov's inequality we have that (a) is satisfied with probability at least $1/2$. For (b), take a pair of vertices u, v as above. Let $\mathcal{F}_{u,v}$ be a family of at least $4\rho n$ internally disjoint paths from u to v , as provided by Claim D. The internal vertices of one path in $\mathcal{F}_{u,v}$ are entirely contained in S with probability ρ^{L+1} . So, the expected number of paths from $\mathcal{F}_{u,v}$ whose internal vertices are entirely contained in S is at least $4\rho n \cdot \rho^{L+1}$. Further, since those paths are internally disjoint, their containment events are independent. Hence, we can apply Theorem 2.9 (i), and get that with probability at least $1 - \exp(-\Theta(n))$, the number of paths from $\mathcal{F}_{u,v}$ whose internal vertices are entirely contained in S is at least $\rho^{L+2}n$. We apply the union bound over the at most n^2 choices of pairs (u, v) to obtain that (b) holds with probability at least $1 - o(1)$. In total, S satisfies all the required properties with positive probability and we can choose such an outcome of the random selection. ◀

Step 4: Constructing an almost spanning system of paths

In this step we find a collection of paths that covers all but at most $\gamma^2 n$ many vertices from X . Throughout the proof, we need to treat the case when $V(\mathcal{P})$ is tiny and when it is substantial slightly differently. We call these two cases ($\heartsuit 1$) and ($\heartsuit 2$).

- *Case ($\heartsuit 1$):* We have $|V(\mathcal{P})| < 2\varepsilon_2 n$.
Set $G_0 := G[V \setminus (V(\mathcal{P}) \cup V(\mathcal{Q}) \cup S)]$. We apply Theorem 2.12 to the graph G_0 with the trivial prepartition $V(G_0) := \emptyset \sqcup V(G_0)$ and with error parameter ε_2 . We obtain an ε_2 -regular partition U_1, \dots, U_{r_2} .
- *Case ($\heartsuit 2$):* We have $|V(\mathcal{P})| \geq 2\varepsilon_2 n$.
Set $G_0 := G[V \setminus (V(\mathcal{Q}) \cup S)]$. We apply Theorem 2.12 to the graph G_0 with a prepartition $V(G_0) := V(\mathcal{P}) \sqcup (V(G_0) \setminus V(\mathcal{P}))$ and with error parameter ε_2 . We obtain an ε_2 -regular partition U_1, \dots, U_{r_2} .

Observe that in both ($\heartsuit 1$) and ($\heartsuit 2$), we have

$$U_i \subseteq V(\mathcal{P}) \quad \text{or} \quad U_i \subseteq V \setminus (V(\mathcal{P}) \cup V(\mathcal{Q}) \cup S) \quad (33)$$

for every $i \in [r_2]$.

Since the cluster sizes differ by at most 1 (c.f. (R1)), we can find an integer k so that for every $i \in [r_2]$ we have

$$2k \leq |U_i| \leq 2k + 2. \quad (34)$$

Let G'_0 be obtained from G_0 by removing all edges $uv \in E(G_0)$ with

- $u, v \in U_i$ for some $i \in [r_2]$, and
- $u \in U_i, v \in U_j$, and (U_i, U_j) is not ε_2 -regular, or
- $u \in U_i, v \in U_j$, and (U_i, U_j) has density smaller than d_2 .

Then by Theorem 2.12, for every vertex $v \in V(G_0)$,

$$\deg_{G_0}(v) < \deg_{G'_0}(v) + 2d_2 n. \quad (35)$$

Define the reduced graph R_2 on $[r_2]$ as before, by putting an edge between i and j if the pair of vertex classes (U_i, U_j) is ε_2 -regular with density at least d_2 .

By (F2) we have $\|W_G - W\|_{\square} \leq \varepsilon_2/2$. Next, we use Lemma 2.5 to find a graphon representation W_{G_0} of G_0 with $\|W_{G_0} - W_G\|_{\square} \leq 5\gamma$, as follows. If we have ($\heartsuit 2$), then we use that $|S \cup V(\mathcal{Q})| \leq 2\gamma n$, and hence there exists a graphon representation W_{G_0} such that $\|W_{G_0} - W_G\|_{\square} \leq 4\gamma$. If we have ($\heartsuit 1$), then we use that $|V(\mathcal{P}) \cup S \cup V(\mathcal{Q})| < 2\gamma n + 2\varepsilon_2 n$, and hence there exists a graphon representation W_{G_0} such that $\|W_{G_0} - W_G\|_{\square} \leq 4\gamma + 4\varepsilon_2$.

By Fact 2.13 there exists a graphon representation W_{R_2} such that $\|W_{R_2} - W_{G_0}\|_{\square} \leq d_2 + 2\varepsilon_2$. Altogether this gives

$$\|W_{R_2} - W\|_{\square} \leq \|W_{R_2} - W_{G_0}\|_{\square} + \|W_{G_0} - W_G\|_{\square} + \|W_G - W\|_{\square} \leq d_2 + 2\varepsilon_2 + 5\gamma + \varepsilon_2/2 < \varepsilon_1.$$

Hence, we can apply Proposition 4.3 with β instead of γ and with $C = \frac{1}{100\beta}$ on R_2 and W . In particular, the subgraph $R_2^* = R_2 - D_{R_2}(\alpha)$ satisfies (a)–(c) for every $A_2 \subseteq V(R_2^*)$ with $|A_2| \leq \alpha r_2/100$. Set

$$A_2 := \{i \in V(R_2^*) : U_i \subseteq V(\mathcal{P})\}.$$

(In $(\heartsuit 1)$, we of course have $A_2 = \emptyset$.) Due to (33) if $i \notin A_2$ then $U_i \subseteq V \setminus (V(\mathcal{Q}) \cup S \cup V(\mathcal{P}))$. We have $|A_2| \leq \frac{|V(\mathcal{P})|}{n} r_2 \stackrel{(F1)(iii)}{\leq} \alpha^2 r_2 \leq \frac{\alpha r_2}{100}$, meaning that the set A_2 satisfies the size condition of Proposition 4.3. Therefore, due to (b), $R_2^* - A_2$ is uniquely half-covered and Lemma 2.4(ii) yields a half-integral perfect matching $m: E(R_2^* - A_2) \rightarrow \{0, \frac{1}{2}, 1\}$. Set $U := \bigcup_{i \in V(R_2^*) \setminus A_2} U_i$. In the claim below, we find an almost-spanning path system in $G_0[U]$.

Claim F. *There exists a collection $\mathcal{F} = \{P_{i,j}\}_{m(ij)>0}$ of vertex-disjoint paths in $G_0[U]$ consisting of at most r_2 paths, covering all except at most ρn many vertices from U .*

Proof of Claim F. For every $i \in V(R_2^*) \setminus A_2$ we find a partition of $U_i = U_{(i,0)} \sqcup \bigsqcup_{j \in V(R_2^*) \setminus (A_2 \cup \{i\})} U_{(i,j)}$, the following way, which depends on how i is covered by m :

- (O1) If there exists $j^* \in V(R_2^*) \setminus (A_2 \cup \{i\})$ such that $m(ij^*) = 1$, then let $U_{(i,j^*)} \subset U_i$ be arbitrary of size $2k$, and for $j \in V(R_2^*) \setminus (A_2 \cup \{i, j^*\})$, set $U_{(i,j)} := \emptyset$.
- (O2) Otherwise (recall that m is perfect and half-integral), there exist two distinct indices $j_1, j_2 \in V(R_2^*) \setminus (A_2 \cup \{i\})$ such that $m(ij_1) = m(ij_2) = 1/2$. Let $U_{(i,j_1)}, U_{(i,j_2)} \subset U_i$ be arbitrary disjoint, each of size k , and set $U_{(i,j)} := \emptyset$ for all other indices j .

In either case, define $U_{(i,0)} := U_i \setminus \bigsqcup_{j \in V(R_2^*) \setminus (A_2 \cup \{i\})} U_{(i,j)}$, and note by (34) that

$$|U_{(i,0)}| \leq 2 \leq \rho |U_i|/2. \quad (36)$$

Consider an arbitrary edge $ij \in E(R_2)$ such that $m(ij) > 0$. Irrespective of whether the pair $(U_{(i,j)}, U_{(j,i)})$ was obtained using (O1) or (O2), it spans at least one third of the pair (U_i, U_j) . Since the pair (U_i, U_j) is ε_2 -regular of density at least d_2 , by Fact 2.11, the pair $(U_{(i,j)}, U_{(j,i)})$ is $3\varepsilon_2$ -regular of density at least $d_2/2$. By Fact 2.16 there is a path $P_{i,j} \subset G[U_{(i,j)}, U_{(j,i)}]$ covering all but at most $\rho |U_{(i,j)}|/2$ many vertices from $U_{(i,j)}$ and all but at most $\rho |U_{(j,i)}|/2$ many vertices from $U_{(j,i)}$. Taking into account that also the vertices of $U_{(i,0)}$ are not covered by paths, we see using (36) that our system of paths covers all but at most $\rho |U_i|$ vertices in each cluster U_i . Summing over all i , we obtain the bound for the uncovered vertices in the Claim. Finally, note that the number edges ij with $m(ij) > 0$ is at most r_2 . For each such edge, one path was added to the system. This gives the bound on the number of paths in the Claim. \blacktriangleleft

Note that the vertices in $\bigcup_{i \in A_2 \cup D_{R_2}(\alpha)} U_i$ are not covered by \mathcal{F} . We shall show that they are covered by $V(\mathcal{P})$.⁴ First note that directly from the definition of A_2 we have $\bigcup_{i \in A_2} U_i \setminus V(\mathcal{P}) = \emptyset$. Secondly, consider an arbitrary $i \in D_{R_2}(\alpha)$ and a vertex $v \in U_i$. Due to (35), we have

$$\deg_G(v) < \alpha n + 2d_2 n < 2\alpha n,$$

which by (F1)(i) means that v was already covered by \mathcal{P} and therefore $v \in V(\mathcal{P})$. Hence,

$$|V \setminus (V(\mathcal{F}) \cup V(\mathcal{P}) \cup V(\mathcal{Q}) \cup S)| = |U \setminus V(\mathcal{F})| + \left| \bigcup_{i \in A_2 \cup D_{R_2}(\alpha)} U_i \setminus V(\mathcal{P}) \right| \leq \rho n. \quad (37)$$

⁴Note that in case $(\heartsuit 1)$, this actually means $\bigcup_{i \in A_2 \cup D_{R_2}(\alpha)} U_i = \emptyset$, which is what the argument below indeed gives.

In other words the path system $\mathcal{P} \cup \mathcal{F} \cup \{Q\}$ covers all of $V \setminus S$ except for at most ρn many vertices, all of them lying in $X \subseteq V \setminus V(\mathcal{P})$.

Step 5: Connecting all paths and completing the cycle

Finally, the path tiling given by $\mathcal{P} \cup \mathcal{F} \cup \{Q\}$ contains at most $2/\alpha + r_2 + 1 \leq 2r_2$ many paths. By construction of \mathcal{F} and Q and by using (F1)(ii) all these paths end in vertices in X . Moreover by Claim E(b), between every two of these vertices there are at least $\rho^{L+2}n/2 > 2r_2$ disjoint paths connecting them, with all internal vertices contained in S . Thus, we may straightforwardly connect all these paths one by one, into a single cycle C , such that all new vertices are taken from S . In other words, $V(C) = V(\mathcal{F}) \cup V(\mathcal{P}) \cup V(Q) \cup S'$, where S' is a suitable subset of S . Due to (37) and Claim E(a), the number of vertices not covered by C is at most

$$\rho n + |S| \leq 3\rho n \leq \gamma^2 n.$$

Since all the uncovered vertices lie in $X \setminus V(\mathcal{P})$, the absorbing property of Q yields a Hamilton cycle in G .

7 Concluding remarks

7.1 Hamilton cycles with positive probability

Theorem 1.2 identifies conditions (I), (II), and (III) on a graphon W which are sufficient for $\mathbb{G}(n, W)$ to be Hamiltonian asymptotically almost surely. Furthermore, parts (A), (B), and (C) demonstrate that these conditions are necessary. We can refine the negative cases. Specifically, we aim to distinguish between graphons for which $\mathbb{G}(n, W)$ is Hamiltonian asymptotically almost never, and those where the property holds with probability bounded away from 0 and 1.

Below, we list four conditions, each of which guarantees that $\mathbb{G}(n, W)$ is a.a.s. not Hamiltonian. We conjecture that this list characterizes all such strongly negative cases.

Proposition 7.1. *Suppose that $W : \Omega^2 \rightarrow [0, 1]$ is a graphon. Then $\mathbb{G}(n, W)$ is asymptotically almost never Hamiltonian if at least one of the following conditions holds:*

- (i) W is not a connected graphon;
- (ii) $\lim_{\alpha \searrow 0} \frac{\mu(D_\alpha(W))}{\alpha} = \infty$;
- (iii) W has a narrow peninsula;
- (iv) there exists a partition $\Omega = S \sqcup T$ with $\mu(S) = \mu(T) = \frac{1}{2}$ such that W is zero almost everywhere on $(S \times S) \cup (T \times T)$.

Conjecture 7.2. *The list in Proposition 7.1 is complete. That is, if a graphon W does not satisfy any of the conditions (i)–(iv), then*

$$\limsup_{n \rightarrow \infty} \mathbf{P}[\mathbb{G}(n, W) \text{ is Hamiltonian}] > 0.$$

We briefly sketch the justification for Proposition 7.1. The validity of condition (i) is discussed in Section 1.2.1. Condition (ii) follows from Theorem 3(a) in [17], which establishes that such graphons yield disconnected random graphs a.a.s.

For condition (iii), the argument is analogous to, but simpler than, the one presented in Section 1.2.3. Suppose (A, B) is a witness that W has a narrow peninsula (recall Definition 1.1(ii)). By the Law of Large Numbers, after the vertex-generating stage (GS1), the partition of the vertex set $V = V_A \sqcup V_B \sqcup V_{\text{rest}}$ (induced by types in A , B , or $\Omega \setminus (A \cup B)$) satisfies $|V_A| = (\mu(A) + o(1))n$ and $|V_B| = (\mu(B) + o(1))n$ a.a.s. Since W vanishes on $A \times (A \cup B)$, in the edge-generating step (GS2), no edges are inserted between V_A and $V_A \cup V_B$ a.a.s. Consider the function $f : V \rightarrow \{0, \frac{1}{2}, 1\}$ defined by

$$f(v) = \begin{cases} 0 & \text{if } v \in V_A, \\ \frac{1}{2} & \text{if } v \in V_B, \\ 1 & \text{if } v \in V_{\text{rest}}. \end{cases}$$

Obviously, this function f constitutes a half-integral vertex cover. Its total weight is

$$\|f\|_1 = \frac{1}{2}|V_B| + |V_{\text{rest}}| = \left(\frac{1}{2}\mu(B) + 1 - \mu(A) - \mu(B) + o(1)\right)n.$$

Using the peninsula parameters $\mu(B) = 1 - 2a$ and $\mu(A) > a$, a straightforward calculation shows that $\|f\|_1 = (\frac{1}{2} - (\mu(A) - a) + o(1))n < \frac{n}{2}$. Consequently, the random graph does not contain a fractional perfect matching (and thus no Hamilton cycle) a.a.s.

Finally, regarding condition (iv), note that $|V_S| \sim \text{Bin}(n, 1/2)$. By the Central Limit Theorem, the probability that $|V_S| = |V_T| = n/2$ tends to 0. Thus, a.a.s., the vertex set contains an independent set (either V_S or V_T) of size strictly greater than $n/2$, and hence is not Hamiltonian.

7.2 Sparser inhomogeneous random graphs

For any graphon $W \not\equiv 0$, the number of edges in $\mathbb{G}(n, W)$ is strongly concentrated around $\frac{1}{2}\|W\|_1 \cdot n^2 = \Theta(n^2)$. Hence, $\mathbb{G}(n, W)$ serves as an inhomogeneous counterpart to the dense Erdős–Rényi random graph $\mathbb{G}(n, p)$ with a constant edge probability $p \in (0, 1]$. We can consider sparser versions $\mathbb{G}(n, f(n) \cdot W)$ for some function $f(n) \rightarrow 0$. In such a setting, W need not be bounded anymore. As the most prominent example, [3] studies the giant component in inhomogeneous random graphs in the sparsest meaningful regime $f(n) = \frac{1}{n}$.

Given the threshold for Hamiltonicity in the homogeneous case obtained by Pósa and Korshunov, it is natural to consider the scaling $f(n) = \frac{\log n}{n}$. The related problem of *connectivity* was solved for this regime in [6] under two additional assumptions. The most notable of these—the continuity of W —is likely a technical requirement of the proof technique rather than a fundamental structural necessity. Namely, [6] shows that for $\mathbb{G}(n, \frac{\log n}{n} \cdot W)$ to be a.a.s. connected, we need condition (C1) alongside a stricter degree requirement: the essential infimum of the degrees in W must be strictly greater than 1. This elegantly contrasts with our condition (C2) from the dense regime, which only requires a light tail of small degrees. The necessity of this hard lower bound in the sparse setting stems directly from the threshold for the disappearance of isolated vertices; at the critical scaling $f(n) = \frac{\log n}{n}$, any positive measure of points with graphon degrees

less than $\frac{\log n}{n}$ would a.a.s. yield isolated vertices. In analogy to Theorem 1.2, we believe that if these two conditions from [6] hold and W does not have a peninsula, then $\mathbb{G}(n, \frac{\log n}{n} \cdot W)$ is a.a.s. Hamiltonian.

7.3 Other spanning structures in $\mathbb{G}(n, W)$

We can study the appearance of other spanning structures than a Hamilton cycle in $\mathbb{G}(n, W)$. Perhaps the closest one is a perfect matching, by which we mean — in order to avoid a parity issue — a matching covering at least $n - 1$ vertices. Are all the conditions on W in Theorem 1.2 necessary for $\mathbb{G}(n, W)$ to contain a.a.s. a perfect matching? Conditions (II) and (III) clearly are, as the features given in (B) and (C) are clear obstructions to the property of having a perfect matching, too.

Less naturally, Condition (I) is also necessary. Indeed, if W is not connected, then we can use a witness of its disconnectedness $\Omega = S \sqcup T$ to partition the vertices of $\mathbb{G}(n, W)$ into two types, $V = V_S \sqcup V_T$, and we know that each edge is contained entirely in V_S or in V_T , a.a.s. The random variable $|V_S|$ has distribution $\text{Bin}(n, \mu(S))$. It is easy to check that when n is even, with probability $\frac{1}{2} + o(1)$, both $|V_S|$ and $|V_T|$ are odd. Obviously, $\mathbb{G}(n, W)$ does not contain a perfect matching in that case.

The next natural spanning structure one might study is the square (or higher powers, say the k -th power) of a Hamilton cycle. Here, even the homogeneous (Erdős–Rényi) case is very complicated, with thresholds determined only partially (for squares of Hamilton cycles in [20], up to a constant, and for $k \geq 4$ in [27]). In any counterpart of Theorem 1.2 for higher powers, we are likely to need to find the correct counterpart only to condition (III). Indeed, it was shown in Theorem 4 in [17] that conditions (I) and (II) in fact guarantee higher vertex connectivity, which we need to have in the presence of a higher power of a Hamilton cycle.

That is, while (III) stems from the fact that a Hamilton cycle contains a perfect matching (i.e., a perfect tiling by copies of K_2), the sought counterpart will be based on the fact that the k -th power of a Hamilton cycle contains a perfect tiling by copies of K_{k+1} .⁵ Paper [15] develops a graphon theory for tilings. Similarly to the graphon theory of matchings (recall the beginning of Section 4.1), the dual approach turns out to be cleaner. In particular, paper [15] introduces the notion of a K_{k+1} -fractional cover of W , which is any measurable function $f : \Omega \rightarrow [0, 1]$ such that for μ^{k+1} -almost every $(k+1)$ -tuple $(x_1, \dots, x_{k+1}) \in \Omega^{k+1}$ we have that $\sum_{i=1}^{k+1} f(x_i) \geq 1$ or $\prod_{i=1}^k \prod_{j=i+1}^{k+1} W(x_i, x_j) = 0$. Paper [15] shows that taking the infimum of $\|f\|_1$ over all K_{k+1} -fractional covers of W leads to a parameter which corresponds to the proportional size of the largest tiling by K_{k+1} 's in graphs close to W (in particular, Theorem 5.1 in [15] is relevant). Hence, our conjecture, whose case $k = 1$ corresponds to Theorem 1.2, is as follows.

Conjecture 7.3. *Suppose that $W : \Omega^2 \rightarrow [0, 1]$ is a graphon, and $k \geq 2$ is fixed. Then $\mathbb{G}(n, W)$ contains a.a.s. the k -th power of a Hamilton cycle if and only if the following three conditions are fulfilled:*

(i) *W is a connected graphon,*

(ii) *we have $\lim_{\alpha \searrow 0} \frac{\mu(D_\alpha(W))}{\alpha} = 0$, and*

(iii) *the only K_{k+1} -fractional cover of W of L^1 -norm at most $\frac{1}{k+1}$ is the constant- $\frac{1}{k+1}$ function.*

⁵A tiling by copies of K_{k+1} is a collection of vertex-disjoint copies of K_{k+1} .

Acknowledgments

The sketch of the beautiful peninsula was generated by Gemini.

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