TILING EDGE-ORDERED GRAPHS WITH MONOTONE PATHS AND OTHER STRUCTURES

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Abstract. Given graphs F and G, a perfect F-tiling in G is a collection of vertex-disjoint copies of F in G that together cover all the vertices in G. The study of the minimum degree threshold forcing a perfect F-tiling in a graph G has a long history, culminating in the Kühn–Osthus theorem [Combinatorica 2009] which resolves this problem, up to an additive constant, for all graphs F. In this paper we initiate the study of the analogous question for edge-ordered graphs. In particular, we characterize for which edge-ordered graphs F this problem is well-defined. We also apply the absorbing method to asymptotically determine the minimum degree threshold for forcing a perfect P-tiling in an edge-ordered graph, where P is any fixed monotone path.

1. Introduction

1.1. **Monotone paths in edge-ordered graphs.** An *edge-ordered graph* G is a graph equipped with a total order $\leq$ of its edge set $E(G)$. Usually we will think of a total order of $E(G)$ as a labeling of the edges with labels from $\mathbb{R}$, where the labels inherit the total order of $\mathbb{R}$ and where edges are assigned distinct labels. A path P in G is *monotone* if the consecutive edges of P form a monotone sequence with respect to $\leq$. We write $P_k^{\leqslant}$ for the monotone path of length k (i.e., on k edges).

The study of monotone paths in edge-ordered graphs dates back to the 1970s. Chvátal and Komlós [7] raised the following question: what is the largest integer $f(K_n)$ such that every edge-ordering of K_n contains a copy of the monotone path $P_{f(K_n)}^{\leqslant}$ of length $f(K_n)$? Over the years there have been several papers on this topic [3, 4, 5, 12, 18, 20]. In a recent breakthrough, Bucić, Kwan, Pokrovskiy, Sudakov, Tran, and Wagner [3] proved that $f(K_n) \geq n^{1-o(1)}$. The best known upper bound on $f(K_n)$ is due to Calderbank, Chung, and Sturtevant [5] who proved that $f(K_n) \leq (1/2 + o(1))n$. There have also been numerous papers on the wider question of the largest integer $f(G)$ such that every edge-ordering of a graph G contains a copy of a monotone path of length $f(G)$. See the introduction of [3] for a detailed overview of the related literature.

A classical result of Rödl [20] yields a Turán-type result for monotone paths: every edge-ordered graph with n vertices and with at least $k(k+1)n/2$ edges contains a copy of $P_k^{\leqslant}$. More recently, Gerbner, Methuku, Nagy, Pálvölgyi, Tardos, and Vizer [11] initiated the systematic study of the Turán problem for edge-ordered graphs.

It is also natural to seek conditions that force an edge-ordered graph G to contain a collection of vertex-disjoint monotone paths $P_k^{\leqslant}$ that cover all the vertices in G, that is, a *perfect $P_k^{\leqslant}$-tiling*

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in G. Our first result asymptotically determines the minimum degree threshold that forces a perfect $P_k^{\leqslant}$-tiling.

Theorem 1.1. *Given any $k \in \mathbb{N}$ and $\eta > 0$, there exists an $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ where $(k+1)|n$ then the following holds: if G is an n-vertex edge-ordered graph with minimum degree*

$$\delta(G) \geq (1/2 + \eta)n$$

then G contains a perfect $P_k^{\leqslant}$-tiling. Moreover, for all $n \in \mathbb{N}$ with $(k+1)|n$, there is an n-vertex edge-ordered graph G_0 with $\delta(G_0) \geq \lfloor n/2 \rfloor - 2$ that does not contain a perfect $P_k^{\leqslant}$-tiling.

The proof of Theorem 1.1 provides the first application of the so-called *absorbing method* in the setting of edge-ordered graphs.

1.2. **The general problem.** Let F and G be edge-ordered graphs. We say that G *contains* F if F is isomorphic to a subgraph F' of G; here, crucially, the total order of $E(F)$ must be the same as the total order of $E(F')$ that is inherited from the total order of $E(G)$. In this case we say F' is a *copy of F in G*. For example, if G contains a path F' of length 3 with consecutive edges labeled 5, 17 and 4 then F' is a copy of the path F of length 3 with consecutive edges labeled 2, 3 and 1.

Given edge-ordered graphs F and G, an *F-tiling* in G is a collection of vertex-disjoint copies of F in G; an F-tiling in G is *perfect* if it covers all the vertices in G. In light of Theorem 1.1 we raise the following general question.

Question 1.2. *Let F be a fixed edge-ordered graph on $f \in \mathbb{N}$ vertices and let $n \in \mathbb{N}$ be divisible by f. What is the smallest integer $f(n, F)$ such that every edge-ordered graph on n vertices and of minimum degree at least $f(n, F)$ contains a perfect F-tiling?*

Theorem 1.1 implies that $f(n, P_k^{\leqslant}) = (1/2 + o(1))n$ for all $k \in \mathbb{N}$. Note that the *unordered* version of Question 1.2 had been well-studied since the 1960s (see, e.g., [1, 8, 13, 15, 16]) and forty-five years later a complete solution, up to an additive constant term, was obtained via a theorem of Kühn and Osthus [16]. Very recently, the *vertex-ordered graph* version of this problem has been asymptotically resolved [2, 10].

Question 1.2 has a rather different flavor to its graph and vertex-ordered graph counterparts. In particular, there are edge-ordered graphs F for which, given *any* $n \in \mathbb{N}$, there exists an edge-ordering $\leq$ of the complete graph K_n that does not contain a copy of F. Thus, for such F, Question 1.2 is trivial in the sense that clearly there is no minimum degree threshold $f(n, F)$ for forcing a perfect F-tiling. This motivates Definitions 1.3 and 1.4 below.

Definition 1.3 (Turánable)**.** An edge-ordered graph F is *Turánable* if there exists a $t \in \mathbb{N}$ such every edge-ordering of the graph K_t contains a copy of F.

An unpublished result of Leeb (see, e.g., [11, 19]) characterizes all those edge-ordered graphs F that are Turánable. Moreover, a result of Gerbner, Methuku, Nagy, Pálvölgyi, Tardos, and Vizer [11, Theorem 2.3] shows that the so-called *order chromatic number* is the parameter that governs the Turán threshold for Turánable edge-ordered graphs F.

Definition 1.4 (Tileable)**.** An edge-ordered graph F on f vertices is *tileable* if there exists a $t \in \mathbb{N}$ divisible by f such that every edge-ordering of the graph K_t contains a perfect F-tiling.

Let F be a tileable edge-ordered graph on f vertices and let $T(F)$ be the smallest possible choice of $t \in \mathbb{N}$ in Definition 1.4 for F. It is easy to see that every edge-ordering of the graph K_s contains a perfect F-tiling for every $s \geq T(F)$ that is divisible by f. Note that Theorem 1.1 implies that $P_k^{\leqslant}$ is tileable for all $k \in \mathbb{N}$.

The second objective of this paper is to provide a characterization of those edge-ordered graphs that are tileable; see Theorem 2.6. Thus, this characterizes for which edge-ordered graphs F Question 1.2 is well-defined.

Interestingly, Theorem 2.6 implies that there are edge-ordered graphs that are Turánable but not tileable; see Proposition 2.10. The precise characterization of the tileable edge-ordered graphs is a little involved, and depends on twenty edge-orderings of K_f; as such, we defer the statement of Theorem 2.6 to Section 2. In [11] it is proven that no edge-ordering of K_4 is Turánable and consequently, any edge-ordered graph containing a copy of K_4 is not Turánable and therefore not tileable. Thus, for an edge-ordered graph to be tileable it cannot be too 'dense'. Here we prove that no edge-ordering of K_4^- is tileable[1]; see Proposition 2.11. However, we prove that the property of being tileable is not closed under subgraphs and there are in fact connected tileable edge-ordered graphs that contain copies of K_4^- (see Corollary 2.15).

A graph H is *universally tileable* if for any given ordering of $E(H)$, the resulting edge-ordered graph is tileable. Similarly, we say that H is *universally Turánable* if given any edge-ordering of H, the resulting edge-ordered graph is Turánable. Using [11, Theorem 2.18] it is easy to characterize those graphs H that are universally tileable.

Theorem 1.5. *Let H be a graph. The following are equivalent:*

(*a*) *H is universally tileable;*
(*b*) *H is universally Turánable;*
(*c*) (*i*) *H is a star forest (possibly with isolated vertices),*[2] *or*
  (*ii*) *H is a path on three edges together with a (possibly empty) collection of isolated vertices, or*
  (*iii*) *H is a copy of K_3 together with a (possibly empty) collection of isolated vertices.*

In Section 3 we determine the asymptotic value of $f(n, F)$ for all connected universally tileable edge-ordered graphs F.

Our characterization of tileable edge-ordered graphs lays the ground for the systematic study of Question 1.2. The second and third authors will investigate this problem further in a forthcoming paper. Already though we can say something about this question. Indeed, an almost immediate consequence of the Hajnal–Szemerédi theorem [13] is the following result.

Theorem 1.6. *Let F be a tileable edge-ordered graph and let $T(F)$ be the smallest possible choice of $t \in \mathbb{N}$ in Definition 1.4 for F. Given any integer $n \geq T(F)$ divisible by $|F|$,*

$$f(n, F) \leq \left(1 - \frac{1}{T(F)}\right) n.$$

The paper is organized as follows. In Section 2.1 we state the characterization of all tileable edge-ordered graphs (Theorem 2.6). Then, in Section 2.2 we use this theorem to provide some basic properties of the family of tileable edge-ordered graphs and some general examples. We give the proof of Theorem 2.6 in Section 2.3. In Section 3 we consider universally tileable graphs, and give the proof of Theorem 1.5. The proof of Theorem 1.6 is given in Section 4. In Section 5 we give the proof of Theorem 1.1. Finally, some concluding remarks are made in Section 6.

Notation. Let G be an (edge-ordered) graph. We write $V(G)$ and $E(G)$ for its vertex and edge sets respectively. We denote an edge $\{u, v\} \in E(G)$ by uv, omitting parenthesis and commas. Define $|G| := |V(G)|$. Given some $X \subseteq V(G)$, we write $G[X]$ for the induced (edge-ordered) subgraph of G with vertex set X. Define $G \setminus X := G[V(G) \setminus X]$. Given $x \in V(G)$ we define $G - x := G[V(G) \setminus \{x\}]$. We define $N_G(x)$ be the set of vertices adjacent to x in G and set $d_G(x) := |N_G(x)|$. When the graph G is clear from the context, we will omit the subscript G here. We say an edge e_1 in G is *larger* than another edge e_2 if e_2 occurs before e_1 in the total order of $E(G)$; in this case we may

[1] Recall that K_t^- denotes the graph obtained from K_t by removing an edge.
[2] A *star forest* is a graph whose components are all stars.

write $e_1 > e_2$ or $e_2 < e_1$. We define *smaller* analogously. A sequence $\{e_i\}_{i\in[t]}$ of edges is *monotone* if $e_1 < e_2 < \cdots < e_t$ or $e_1 > e_2 > \cdots > e_t$.

Given an (unordered) graph G we write $G^{\leqslant}$ to denote the edge-ordered graph obtained from G by equipping $E(G)$ with a total order $\leqslant$. We say that G is the *underlying graph of* $G^{\leqslant}$. Given a graph G together with an (injective) labeling $L : E(G) \to \mathbb{R}$ of its edges, we define the *edge-ordering induced by the labeling* L so that $e_i < e_j$ if and only if $L(e_i) < L(e_j)$. As such, L gives rise to an edge-ordered graph. Note that two *different* labelings can give rise to the *same* edge-ordered graph. For example, a path whose edges are labeled 1, 2, and 3 respectively is a monotone path; likewise a path whose edges are labeled 1, e, and π respectively is a monotone path.

We denote the (unordered) path of length k by P_k and sometimes we identify a copy of P_k with its sequence of vertices $v_1 \cdots v_{k+1}$ where $v_i v_{i+1} \in E(P_k)$ for all $i \in [k]$. Given distinct $a_1, \dots, a_t \in \mathbb{R}$ we write $a_1 \dots a_t$ for the edge-ordered path on t edges whose ith edge has label a_i. For example, $P = 132$ is the edge-ordered path on four vertices v_1, v_2, v_3, v_4 whose first edge v_1v_2 is labeled 1, second edge v_2v_3 is labeled 3, and third edge v_3v_4 is labeled 2.

Given $k \in \mathbb{N}$ and a set X, we write $\binom{X}{k}$ for the collection of all subsets of X of size k.

2. The characterization of all tileable edge-ordered graphs

2.1. **The characterization theorem.** The following Ramsey-type result, attributed to Leeb (see [11, 19]), says that in every sufficiently large edge-ordered complete graph we must always find a subgraph which is *canonically ordered* (see Definition 2.2). Before giving the precise description of the canonical orderings, let us present Leeb's result.

Proposition 2.1. *For every $k \in \mathbb{N}$ there is an $m \in \mathbb{N}$ such that every edge-ordered complete graph K_m contains a copy of K_k that is canonically edge-ordered.* □

We now define the canonical orderings of K_n.

Definition 2.2. Given $n \in \mathbb{N}$, we denote by $\{v_1, \dots, v_n\}$ the vertex set of the complete graph K_n. The following labelings L_1, L_2, L_3, and L_4 induce the *canonical orderings* of K_n.

- *min ordering*: For $1 \le i < j \le n$ the label of the edge v_iv_j is $L_1(v_iv_j) = 2ni + j - 1$.
- *max ordering*: For $1 \le i < j \le n$ the label of the edge v_iv_j is $L_2(v_iv_j) = (2n-1)j + i$.
- *inverse min ordering*: For $1 \le i < j \le n$ the label of the edge v_iv_j is $L_3(v_iv_j) = (2n+1)i - j$.
- *inverse max ordering*: For $1 \le i < j \le n$ the label of the edge v_iv_j is $L_4(v_iv_j) = 2nj - i + n$.

We say that min, max, inverse min, and inverse max are *types* of canonical orderings and that the labelings L_1, L_2, L_3, and L_4 are the *standard labelings* for those types.[3] To emphasize, in the statement of Proposition 2.1, by 'a copy of K_k that is canonically edge-ordered', we mean that the edge-ordering of K_k is the same as the edge-ordering induced by the labeling L_i, for some $i \in [4]$.

Observe that the max and inverse max orderings are the 'reverse' of the min and inverse min orderings respectively. For example, if you reverse the total order of $E(K_n)$ induced by the min ordering L_1, then you obtain an edge-ordered graph whose total order is now induced by the max ordering L_2; here though vertex v_n is playing the role of v_1, v_{n-1} is playing the role of v_2, etc.

Remark 2.3. Whilst the standard labelings formally define the canonical orderings, recalling the following intuitive explanations of these orderings will aid the reader throughout the paper:

[3] The labelings L_1, L_2, L_3, and L_4 presented here differ from those used in [11]. However, the induced edge-orderings are the same. This labeling will be useful for Definition 2.7.

- *min ordering*: the smallest edges are incident to v_1 so that $v_1v_2 < \cdots < v_1v_n$; the next smallest edges are those that go from v_2 to the 'right' of v_2 so that $v_2v_3 < \cdots < v_2v_n$; the next smallest edges are those that go from v_3 to the 'right' of v_3 so that $v_3v_4 < \cdots < v_3v_n$, and so forth.
- *max ordering*: the largest edges are incident to v_n so that $v_1v_n < \cdots < v_{n-1}v_n$; the next largest edges are those that go from v_{n-1} to the 'left' of v_{n-1} so that $v_1v_{n-1} < \cdots < v_{n-2}v_{n-1}$, and so forth.
- *inverse min ordering*: the smallest edges are incident to v_1 so that $v_1v_n < \cdots < v_1v_2$; the next smallest edges are those that go from v_2 to the 'right' of v_2 so that $v_2v_n < \cdots < v_2v_3$, and so forth.
- *inverse max ordering*: the largest edges are incident to v_n so that $v_1v_n > \cdots > v_{n-1}v_n$; the next largest edges are those that go from v_{n-1} to the 'left' of v_{n-1} so that $v_1v_{n-1} > \cdots > v_{n-2}v_{n-1}$, and so forth.

In [11] it was observed that Proposition 2.1 yields a full characterization of Turánable graphs.

Theorem 2.4 (Turánable characterization)**.** *An edge-ordered graph F on f vertices is Turánable if and only if all four canonical edge-orderings of K_f contain a copy of F.* □

In [11, Theorem 2.5] they present a 'family' version of Theorem 2.4, which implies that F is Turánable if and only if F is contained in every canonical edge-ordering of K_n, *for all* $n \in \mathbb{N}$. However, Theorem 2.4 can be deduced easily from the following fact.

Fact 2.5. *Suppose $k \leq n$ are positive integers. If K_n is canonically edge-ordered, then $K_k \subseteq K_n$ is canonically edge-ordered. Moreover, K_k has the same type of canonical edge-ordering as K_n.* □

The picture is slightly different when one seeks a perfect F-tiling instead of just a single copy of F. To illustrate, consider a canonical ordering of K_n with an extra 'defective' vertex x, whose edges incident to it can have an arbitrary ordering. To have a perfect F-tiling in this edge-ordered graph, there must be a copy of F containing the vertex x. This leads to a generalization of the canonical orderings above, which we call $\star$*-canonical orderings* (see Definition 2.7). We obtain a similar characterization for tileable graphs as follows.

Theorem 2.6 (Tileable characterization)**.** *An edge-ordered graph F on f vertices is tileable if and only if all twenty $\star$-canonical orderings of K_f contain a copy of F.*

To define the $\star$-canonical orderings we will consider an edge-ordering of the complete graph K_{n+1} for which there is a vertex $x \in V(K_{n+1})$ such that $K_{n+1} - x$ is canonically ordered. Depending on the type of canonical ordering and the ordering of the edges incident to x we have, for all $n \geq 4$, twenty possible $\star$-canonical orderings of K_{n+1}.

Definition 2.7. Let $\{x, v_1, \ldots, v_n\}$ denote the vertex set of K_{n+1}. Suppose $L : E(K_{n+1}) \to \mathbb{R}$ is a labeling of the edges of K_{n+1} such that its restriction to $K_{n+1} - x$ is canonical with one of the standard labelings L_1, L_2, L_3, or L_4. Moreover, suppose that the labels $x_i := L(xv_i)$ for $i \in [n]$ satisfy one of the following:

- *Larger increasing orderings*: $x_n > \cdots > x_2 > x_1 > \max_{i<j}\{L(v_iv_j)\}$.
- *Larger decreasing orderings*: $x_1 > x_2 > \cdots > x_n > \max_{i<j}\{L(v_iv_j)\}$.
- *Smaller increasing orderings*: $x_1 < x_2 < \cdots < x_n < \min_{i<j}\{L(v_iv_j)\}$.
- *Smaller decreasing orderings*: $x_n < \cdots < x_2 < x_1 < \min_{i<j}\{L(v_iv_j)\}$.
- *Middle increasing orderings*: $x_i = 2ni$ for all $i \in [n]$.

Then, L induces a $\star$*-canonical ordering* of K_{n+1}. We refer to the vertex x as *the special vertex.*

Observe that depending on the type of canonical ordering of $K_{n+1}-x$ there are four possible larger increasing orderings, larger decreasing orderings, smaller increasing orderings, smaller decreasing orderings and middle increasing orderings. We will refer to these twenty possible cases as *types* of $\star$-canonical orderings. Moreover, we will say that $K_{n+1}-x$ is the *canonical part* of the $\star$-canonical ordering. We sometimes refer to the eight smaller increasing/decreasing orderings as the *smaller orderings*. We define the *larger orderings*, *increasing orderings*, and *decreasing orderings* analogously.

Remark 2.8. In contrast with the other types, in the four middle increasing orderings, the edges incident to the special vertex x are 'in between' the edges of the canonical ordering of $K_{n+1}-x$. More precisely, we have:

- If $K_{n+1}-x$ is a min ordering then $v_{i-1}v_n < xv_i < v_iv_{i+1}$ for every $2 \le i \le n-1$. Additionally, $xv_1 < v_1v_2$ and $v_{n-1}v_n < xv_n$.
- If $K_{n+1}-x$ is a max ordering then $v_{i-1}v_i < xv_i < v_1v_{i+1}$ for every $2 \le i \le n-1$. Additionally, $xv_1 < v_1v_2$ and $v_{n-1}v_n < xv_n$.
- If $K_{n+1}-x$ is an inverse min ordering then $v_iv_{i+1} < xv_i < v_{i+1}v_n$ for every $1 \le i \le n-2$. Additionally, $v_{n-1}v_n < xv_{n-1} < xv_n$.
- If $K_{n+1}-x$ is an inverse max ordering then $v_1v_{i-1} < xv_i < v_{i-1}v_i$ for every $3 \le i \le n$. Additionally, $xv_1 < xv_2 < v_1v_2$.

It is not hard to check that canonical orderings are $\star$-canonical orderings. In particular, a min ordering is a smaller increasing ordering, a max ordering is a larger increasing ordering, an inverse min ordering is a smaller decreasing ordering, and an inverse max ordering is a larger decreasing ordering. In each case, the special vertex x plays the role of either the first or the last vertex in the canonical ordering.

The proof of the 'forwards direction' of Theorem 2.6 relies on the following fact for $\star$-canonical orderings, analogous to Fact 2.5 for canonical orderings.

Fact 2.9. *Suppose $k \le n$ are positive integers. If K_{n+1} is $\star$-canonically edge-ordered with special vertex x, then every subgraph $K_k \subseteq K_{n+1}$ with $x \in V(K_k)$ is $\star$-canonically edge-ordered with the same type as K_{n+1}.*[4] □

The forwards direction of Theorem 2.6 follows easily from this fact. Indeed, if F is tileable, by definition there is some $n \in \mathbb{N}$ so that in any $\star$-canonical ordering of K_{n+1} there is a perfect F-tiling. Fact 2.9 implies that in such a perfect F-tiling there is a copy F' of F which covers x and where $K_{n+1}[V(F')]$ is $\star$-canonically edge-ordered with the same type as K_{n+1}. Thus, this implies that every $\star$-canonical ordering of K_f contains a copy of F.

The proof of the backwards direction of Theorem 2.6 makes use of an approach analogous to that of Caro [6]. More precisely, the intuition is as follows. Choose $t \in \mathbb{N}$ to be sufficiently large compared to f. Recall that due to Proposition 2.1, in any edge-ordering of a sufficiently large K_{n_0} one must find a canonical copy of K_t. Now consider any edge-ordering of K_n where n is much larger than n_0. We may repeatedly find vertex-disjoint copies of a canonical copy of K_t in K_n until we have fewer than n_0 vertices remaining. That is, we have tiled the vast majority of K_n with canonical copies of K_t. The idea is now to incorporate the currently uncovered vertices into these canonical K_t and then split each such 'tile' into many $\star$-canonically edge-ordered copies of K_f. Therefore, the resulting substructure in K_n is a perfect tiling of $\star$-canonically edge-ordered copies of K_f. Now by the choice of F, each such copy of K_f contains a spanning copy of F. Thus, K_n contains a perfect F-tiling, as desired.

[4]Note that it follows from Fact 2.5 that every subgraph $K_k \subseteq K_{n+1}$ with $x \notin V(K_k)$ is canonically ordered of the same type as $K_{n+1}-x$.

We defer the formal proof of Theorem 2.6 to Section 2.3. In the following subsection we will see some applications of Theorems 2.4 and 2.6 to study some properties of the families of Turánable and tileable graphs. In particular, in Proposition 2.10 we apply Theorem 2.6 to prove that the notions of tileable and Turánable are genuinely different. More precisely, we provide an infinite family of Turánable edge-ordered graphs that are not tileable.

2.2. **Turánable and tileable graphs.** Given an edge-ordered graph F we define the *reverse of* F, denoted by $\overleftarrow{F}$, as the same graph but in which all relations in the total order of the edges of F are reversed. More precisely, for $F = (V, E)$ we have $\overleftarrow{F} = (V, E)$ and for every $e_1, e_2 \in E$ we have $e_1 \leq_{\overleftarrow{F}} e_2$ if and only if $e_2 \leq_F e_1$, where $\leq_F$ and $\leq_{\overleftarrow{F}}$ are the total orders of F and $\overleftarrow{F}$ respectively. It is easy to see that F is Turánable if and only if $\overleftarrow{F}$ is Turánable. Indeed, let F be a Turánable edge-ordered graph and consider any edge-ordered copy of K_t, where $t \in \mathbb{N}$ is given by Definition 1.3. Then $\overleftarrow{K_t}$ contains a copy of F, and hence, K_t contains a copy of $\overleftarrow{F}$; thus, $\overleftarrow{F}$ is Turánable. The same argument shows that F is tileable if and only if $\overleftarrow{F}$ is tileable.

Throughout this subsection v_i will denote the ith vertex in a canonical ordering and x will denote the special vertex of a $\star$-canonical ordering. Given edge-ordered graphs F and H, we say that a map $\varphi : V(F) \longrightarrow V(H)$ is an *embedding of* F *into* H if and only if

- φ is injective,
- for every edge $uv \in E(F)$ we have $\varphi(u)\varphi(v) \in E(H)$, and
- for every two edges $uv, wz \in E(F)$ such that $uv < wz$ in the total order of $E(F)$, we have $\varphi(u)\varphi(v) < \varphi(w)\varphi(z)$ in the total order of $E(H)$.

Observe that the fact that H contains a copy of F means there is an embedding from F into H. When the embedding φ is clear from the context we do not explicitly state it, and we simply write $u \mapsto v$ instead of $\varphi(u) = v$.

We now present a Turánable graph that is not tileable. Consider the edge-ordered graph D_n defined in [11] as a graph on vertices $u_1, \dots, u_n$ containing all edges incident to u_1 or u_n. The edges are ordered as $u_1u_2 < u_1u_3 < \cdots < u_1u_n < u_2u_n < \cdots < u_{n-1}u_n$.

Proposition 2.10. *Let $n \geq 4$. Then D_n is Turánable but is not tileable.*

Proof. The fact that D_n is Turánable for every $n \geq 4$ was proven in [11, Proposition 2.12], so we only need to show that it is not tileable.

We prove it is impossible to embed D_n into a $\star$-canonically edge-ordered K_n of type larger decreasing whose canonical part is a min ordering. Let $\{x, v_1, \dots, v_{n-1}\}$ be the vertices of such a $\star$-canonical ordering of K_n with special vertex x. Assume for a contradiction that there is an embedding of D_n into this edge-ordered K_n. Suppose first that the vertex u_1 is embedded onto the special vertex x. Then, there are vertices $v_i, v_j \in V(K_n)$ such that in our embedding we have

$$u_k \mapsto v_i \qquad \text{and} \qquad u_n \mapsto v_j\,,$$

for some $k \in [n-1] \setminus \{1\}$. This immediately yields a contradiction since $u_1u_k < u_ku_n$ in D_n whilst in this type of $\star$-canonical ordering $xv_i > v_iv_j$ for every distinct $i, j \in [n-1]$.

Suppose now that u_i is embedded onto the special vertex x where $i \in [n-1] \setminus \{1\}$. Then there are vertices $v_j, v_k, v_\ell \in V(K_n)$ such that

$$u_1 \mapsto v_j\,, \qquad u_m \mapsto v_k\,, \qquad \text{and} \qquad u_n \mapsto v_\ell\,,$$

for some $m \in [n-1] \setminus \{1\}$. Similarly to before, this yields a contradiction because $u_1u_i < u_mu_n$ while in this type of $\star$-canonical ordering we have $xv_j > v_kv_\ell$ for every distinct $j, k, \ell \in [n-1]$.

The only remaining case is when u_n is embedded onto the special vertex x. Thus, the edges $u_1u_n < u_2u_n < \cdots < u_{n-1}u_n$ are embedded onto the edges of the form v_ix for $i \in [n-1]$. In fact, since the

$\star$-canonical ordering is larger decreasing, we must have that

$$u_i \mapsto v_{n-i} \qquad \text{for every } i \in [n-1]\,.$$

However, this yields a contradiction; indeed, while we have that $u_1u_2 < u_1u_3$ in D_n we have $v_{n-1}v_{n-2} > v_{n-1}v_{n-3}$ in the $\star$-canonically edge-ordered K_n. □

We use Proposition 2.10 to prove that there is no tileable edge-ordering of K_4^-.

Proposition 2.11. *No edge-ordering of K_4^- is tileable.*

Proof. To prove the proposition we will show that the only Turánable edge-ordering of K_4^- is in fact D_4, which, due to Proposition 2.10 is not tileable.

As stated in [11, Section 5], the only Turánable edge-ordering of C_4 with vertices $\{w_1, w_2, w_3, w_4\}$ is given by $w_1w_2 < w_2w_3 < w_1w_4 < w_3w_4$; we denote this edge-ordered graph by C_4^{1243}. Thus, in any Turánable edge-ordering of K_4^- the underlying C_4 must be a copy of C_4^{1243}. Starting with such a copy of C_4^{1243} we obtain a K_4^- by either adding the edge w_1w_3 or w_2w_4.

Take an embedding of C_4^{1243} into the inverse min canonical ordering of K_4 given by

$$w_1 \mapsto v_{i_1}\,, \quad w_2 \mapsto v_{i_2}\,, \quad w_3 \mapsto v_{i_3}\,, \quad \text{and} \quad w_4 \mapsto v_{i_4}\,.$$

We first show that this embedding is unique and given by (2.2) below. Suppose that the edge w_1w_2 is not embedded onto an edge containing $v_1 \in V(K_n)$; in other words, $i_1 \neq 1$ and $i_2 \neq 1$. Thus, there is a $j \in \{2,3,4\}$ such that $v_1v_j = v_{i_3}v_{i_4}$. This is a contradiction, since $v_{i_1}v_{i_2} > v_1v_j$ in the inverse min canonical ordering, while $w_1w_2 < w_3w_4$ in C_4^{1243}. Hence, we have that either $i_1 = 1$ or $i_2 = 1$. In the former case, since $w_2w_3 < w_1w_4$ then we have $v_{i_2}v_{i_3} < v_{i_1}v_{i_4} = v_1v_{i_4}$. But this is a contradiction, because in the inverse min canonical ordering all edges containing v_1 are smaller than the edges not containing it. Therefore, we must have that $i_2 = 1$. Further, observe that $w_1w_2 < w_2w_3$ means that $v_1v_{i_1} < v_1v_{i_3}$, which in the inverse min ordering means that

$$i_3 < i_1\,. \tag{2.1}$$

Since $i_1 \leq 4$ and $i_2 = 1$, we have $2 \leq i_3 \leq 3$. Finally, observe that if $i_3 = 2$, then we have $v_{i_1}v_{i_4} = v_3v_4$. But this is again a contradiction, since v_3v_4 is the largest edge in the inverse min ordering of K_4 while $w_1w_4 < w_3w_4$. Thus we get $i_3 = 3$, which together with (2.1), implies that $i_1 = 4$. Summarizing, we have $i_2 = 1$, $i_3 = 3$ and $i_1 = 4$, which finally gives the embedding

$$w_1 \mapsto v_4\,, \quad w_2 \mapsto v_1\,, \quad w_3 \mapsto v_3\,, \quad \text{and} \quad w_4 \mapsto v_2\,. \tag{2.2}$$

Thus, any Turánable edge-ordering of K_4^- obtained by adding one edge to C_4^{1243} must be embedded into the inverse min canonical ordering of K_4 via (2.2). In this way, after adding the edge w_2w_4 or w_1w_3 to C_4^{1243}, the embedding (2.2) gives rise to the following edge-orderings of K_4^-:

$$w_1w_2 < w_2w_3 < w_2w_4 < w_1w_4 < w_3w_4 \qquad \text{and} \tag{2.3}$$

$$w_1w_2 < w_2w_3 < w_1w_4 < w_3w_4 < w_1w_3\,, \tag{2.4}$$

respectively. The ordering (2.3) corresponds with the edge-ordering of D_4, by taking $u_1 = w_2$, $u_2 = w_1$, $u_3 = w_3$, and $u_4 = w_4$ (see the definition of D_4 before Proposition 2.10).

For (2.4), we shall prove that such an edge-ordering of K_4^- cannot be embedded into the inverse max canonical ordering of K_4, and therefore, it is not Turánable. More precisely, we show that C_4^{1243} has only one possible embedding into the inverse max ordering of K_4, but the embedding of the edge w_1w_3 will lie in a different 'position' than the one given by (2.4).

Let w_1', w_2', w_3', w_4' be the vertices of $\overleftarrow{C}_4^{1243}$, the reverse ordering of C_4^{1243}, with edges

$$w_1'w_2' > w_2'w_3' > w_1'w_4' > w_3'w_4'\,.$$

Here we now denote $\overleftarrow{C}_4^{1243}$ by C_4^{4312}. Recall that the inverse max ordering of K_4 with vertices $\{v_1', v_2', v_3', v_4'\}$ corresponds with the reverse of the inverse min ordering on $\{v_1, v_2, v_3, v_4\}$ by relabeling the vertices as $v_1' = v_4$, $v_2' = v_3$, $v_3' = v_2$, and $v_4' = v_1$. Applying the symmetric reasoning as the one above, we have that there is only one possible embedding of C_4^{4312} into the inverse max ordering of K_4. Namely,

$$w_1' \mapsto v_1', \quad w_2' \mapsto v_4', \quad w_3' \mapsto v_2', \quad \text{and} \quad w_4' \mapsto v_3'. \tag{2.5}$$

Moreover, notice that C_4^{1243} is isomorphic to C_4^{4312} by taking $w_1 = w_3'$, $w_2 = w_4'$, $w_3 = w_1'$, and $w_4 = w_2'$, where w_1, w_2, w_3, w_4 are the vertices of C_4^{1243} as in the beginning of the proof. Thus, an embedding of C_4^{1243} into an inverse max ordering of K_4 must follow (2.5) via this isomorphism to C_4^{4312}. This corresponds to

$$w_1 \mapsto v_2', \quad w_2 \mapsto v_3', \quad w_3 \mapsto v_1' \quad \text{and} \quad w_4 \mapsto v_4'.$$

Finally, the edge w_1w_3 is embedded in this way onto $v_1'v_2'$, which is the smallest edge of the inverse max ordering. In other words we obtain,

$$w_1w_3 < w_1w_2 < w_2w_3 < w_1w_4 < w_3w_4,$$

which is incompatible with (2.4). □

The following two propositions are useful to generate tileable (or Turánable) graphs by appropriately adding a vertex and an edge to a tileable (or Turánable) graph.

Proposition 2.12. *Let F be a Turánable edge-ordered graph and $v \in V(F)$ a vertex incident to the smallest edge in F. Let F' be the edge-ordered graph obtained from F by adding a new vertex v' and an edge between v and v' smaller than all edges in F. Then F' is Turánable.*

Proof. Let $|F| := f$ and vu be the smallest edge in F. We want to embed F' into each canonical ordering of K_{f+1}.

Observe that for the min ordering, inverse min ordering, and max ordering of K_{f+1} we have

$$v_1v_i < v_iv_j \text{ for every distinct } i, j \in \{2, \ldots, f+1\}. \tag{2.6}$$

For these canonical orderings we use that F is Turánable and Fact 2.5 to embed F into $K_{f+1}\big[\{v_2, \ldots, v_{f+1}\}\big]$ and then we embed v' onto v_1. Let $i, j \geq 2$ be such that v and u are embedded in this way onto the vertices v_i and v_j respectively. Since vu is the minimal edge in F, then v_iv_j is minimal in our embedding of F into $K_{f+1}\big[\{v_2, \ldots, v_{f+1}\}\big]$. Thus, since $v' \mapsto v_1$ and $v_1v_i < v_iv_j$ by (2.6), this embedding gives rise to a copy of F' in these canonical edge-orderings of K_{f+1}.

For the inverse max ordering, we proceed as follows. Let $t \in [f]$ be such that there is an embedding of F into an inverse max ordering of K_f where v_t plays the role of v. Since F is Turánable and due to Fact 2.5, we can embed F into $K_{f+1}[\{v_1, \ldots, v_{t-1}, v_{t+1}, \ldots, v_{f+1}\}]$ with v_{t+1} playing the role of v. We extend this embedding by assigning v' to v_t. In this way $v'v$ is mapped to v_tv_{t+1} and uv is mapped to an edge of the form $v_{t+1}v_i$ for $i \neq t$. By the definition of the inverse max ordering we have $v_tv_{t+1} < v_{t+1}v_i$, i.e., the embedding of the edge vv' is smaller than the embedding of the edge uv. Thus, the inverse max ordering of K_{f+1} contains a copy of F'. □

Proposition 2.13. *Let F be a tileable edge-ordered graph and $v \in V(F)$ a vertex incident to the smallest edge in F. Let F' be the edge-ordered graph obtained from F by adding a new vertex v' and an edge between v and v' smaller than all edges in F. Then F' is tileable.*

Proof. Let $|F| := f$ and uv be the smallest edge in F. We want to embed F' into each $\star$-canonical ordering of K_{f+1}. We divide the proof into cases depending on the type of the $\star$-canonical ordering.

For smaller orderings of K_{f+1}, we use that F is Turánable to first embed F into a canonical ordering of the same type as the canonical part $K_{f+1} - x$. We then extend this embedding by setting $v' \mapsto x$. The edge vv' is embedded onto an edge of the form xv_j with $j \in [f]$. Thus, by definition of the smaller orderings, our embedding corresponds to a copy of F' in K_{f+1}.

In fact, in the argument above we only used that the smaller orderings satisfy

$$xv_i < v_iv_j \text{ for every distinct } i, j \in [f]\,, \tag{2.7}$$

since we only need that the embedding of vv' is smaller than the embedding of the smallest edge in F. More precisely, observe that if $v' \mapsto x$ and $v \mapsto v_i$ for some $i \in [f]$, then the edge vv' in F' is sent to the edge xv_i and the minimal edge of F, uv, is sent to an edge of the form v_iv_j in K_{f+1} for a $j \in [f] \setminus \{i\}$. Thus, if (2.7) holds, then the embedding of vv' is smaller than embedding of the smallest edge in F, yielding a copy of F'. It is easy to check that (2.7) holds for a middle increasing ordering whose canonical part is an inverse max ordering. Indeed, following the labelings in Definitions 2.2 and 2.7, for a middle increasing ordering whose canonical part is an inverse max ordering we have

$$L_4(xv_i) = 2fi < 2fj - i + f = L_4(v_iv_j) \quad \text{for } 1 \le i < j \le f\,, \quad \text{and}$$
$$L_4(xv_i) = 2fi < 2fi - j + f = L_4(v_iv_j) \quad \text{for } 1 \le j < i \le f\,.$$

Thus, for this $\star$-canonical ordering we can proceed as described above.

We shall now address the remaining $\star$-canonical orderings of K_{f+1}, these are: all larger orderings and all middle increasing orderings except when the canonical part is an inverse max ordering. For these $\star$-canonical orderings of K_{f+1}, we will proceed differently depending on how F embeds into a $\star$-canonically edge-ordered K_f of the same type as K_{f+1}. Recall that such embeddings exist due to Fact 2.9 and because F is a tileable edge-ordered graph.

First, note that for all remaining $\star$-canonical orderings

$$v_1v_i < xv_i \quad \text{for every } 2 \le i \le f\,. \tag{2.8}$$

Indeed, for the larger orderings this follows directly from the definition. For the middle increasing orderings whose canonical part is not the inverse max ordering, we just need to check the following inequalities given by the labelings in Definitions 2.2 and 2.7 for $2 \le i \le f$:

- for the canonical part being a min ordering $L_1(v_1v_i) = 2f + i - 1 < 2fi = L_1(v_ix)\,,$
- for the canonical part being a max ordering $L_2(v_1v_i) = (2f-1)i + 1 < 2fi = L_2(v_ix)\,,$ and
- for the canonical part being an inverse min ordering $L_3(v_1v_i) = (2f+1) - i < 2fi = L_3(v_ix)\,.$

Note that (2.8) does not hold for smaller orderings or for middle increasing orderings whose canonical part is an inverse max ordering.

Now suppose that in an embedding of F into a $\star$-canonically edge-ordered K_f of the same type as K_{f+1}, vertex u is embedded as the special vertex x. Then, we embed F' into the $\star$-canonically edge-ordered K_{f+1} by first embedding F into $K_{f+1}[\{x, v_2, \dots, v_f\}]$ with u as the special vertex, and then mapping v' to v_1. To check that this is an embedding of F' into K_{f+1} observe that $v'v$ is embedded onto v_1v_i and uv is embedded onto xv_i, for some $i \ge 2$. Since uv is the smallest edge in F, the edge xv_i is the smallest in our embedding of F into $K_{f+1}[\{x, v_2, \dots, v_f\}]$. Due to (2.8), $v_1v_i < xv_i$, and so the edge $v'v$ is mapped to an edge smaller than all the edges in our copy of F. This yields a copy of F' in K_{f+1}.

Next suppose that in an embedding of F into a $\star$-canonically edge-ordered K_f of the same type as K_{f+1}, v is embedded onto the special vertex x. If the $\star$-canonical ordering is increasing, then we embed F into $K_{f+1}[\{x, v_2, \dots, v_f\}]$ with v as the special vertex, and map v' to v_1. Thus, the edge vv' is embedded onto xv_1 and uv is embedded onto an edge xv_i for some $i \ge 2$. Since the $\star$-canonical ordering is increasing we have $xv_1 < xv_i$. As before this yields an embedding of F' into K_{f+1}. If the $\star$-canonical ordering is decreasing we proceed analogously by first embedding F into $K_{f+1}[\{x, v_1, \dots, v_{f-1}\}]$ and then extending that embedding by assigning v' to v_f.

Finally, suppose that in all embeddings of F into a $\star$-canonically edge-ordered K_f of the same type as K_{f+1}, neither u nor v is embedded as the special vertex x. Then we proceed similarly to the proof of Proposition 2.12 above. If the canonical part is a min ordering, an inverse min ordering, or a max ordering, then we first embed F into $K_{f+1}\big[\{x, v_2, \dots, v_{f+1}\}\big]$ and then v' onto v_1. Let $i, j \geq 2$ be such that v and u are embedded in this way onto vertices v_i and v_j respectively, both in the canonical part of $K_{f+1}\big[\{x, v_2, \dots, v_{f+1}\}\big]$. Then, since (2.6) holds in this context for the edge-ordering of the canonical part, we have that $v_1 v_i < v_i v_j$. Hence, $v'v$ is mapped to an edge, $v_1 v_i$, that is smaller than the edge $v_i v_j$ that uv is mapped to. As before this yields an embedding of F' into K_{f+1}. If the canonical part is an inverse max ordering, let $t \in [f]$ be such that there is an embedding of F into the $\star$-canonically edge-ordered K_f of the same type as K_{f+1} for which $v \mapsto v_t$. Then we embed F into $K_{f+1}[\{x, v_1, \dots, v_{t-1}, v_{t+1}, \dots, v_f\}]$ in such a way that $v \mapsto v_{t+1}$. We are assuming that in every embedding of F into a $\star$-canonically edge-ordered K_f of the same type as K_{f+1}, neither v nor u is embedded as the special vertex x; so there is an $i \in [f] \setminus \{t\}$ such that $u \mapsto v_i$ in our embedding. Extend this embedding by assigning v' to v_t. In this way we have

$$v'v \mapsto v_t v_{t+1} \qquad \text{and} \qquad uv \mapsto v_{t+1} v_i\,.$$

In the inverse max ordering we have $v_t v_{t+1} < v_{t+1} v_i$, which means that the edge $v'v$ is mapped to is smaller than edge uv is mapped to. As before this yields a copy of F' into K_{f+1}. □

Using Theorems 2.4 and 2.6 it is easy to see that any Turánable edge-ordered graph becomes tileable after adding an isolated vertex. More interestingly, the next proposition implies that given any connected Turánable graph F we can obtain a connected tileable graph on $|F| + 2$ vertices.

Given a Turánable edge-ordered graph F on f vertices, we say a vertex $v \in V(F)$ is *minimal* if it plays the role of v_1 in an embedding of F into a min ordering of K_f. Similarly, we say that v is *maximal* if it plays the role of v_f in an embedding of F into a max ordering of K_f. By Theorem 2.4 a Turánable graph always contains at least one minimal and one maximal vertex[5]. Observe that the edges incident to a minimal (resp. maximal) vertex are always smaller (resp. larger) than the edges not incident to it.

We show that starting with a Turánable graph we can add two pendant edges, one to a minimal vertex and one to a maximal vertex, and obtain a tileable graph. This result, together with the example of a Turánable graph D_n that is not tileable (see Proposition 2.10), implies the perhaps surprising property that being tileable is not closed under taking connected subgraphs.

Proposition 2.14. *Let F be an edge-ordered Turánable graph with $\underline{v}, \overline{v} \in V(F)$ being distinct non-isolated minimal and maximal vertices respectively. Let F' be constructed by adding two new vertices $\underline{u}, \overline{u}$ and the edges $\underline{uv}$ and $\overline{uv}$ such that $\underline{uv}$ is smaller than all other edges and $\overline{uv}$ is larger than all other edges. Then F' is tileable.*

Proof. Let $f := |F|$. As F is Turánable, by Proposition 2.12 we have that $F' - \overline{u}$ is Turánable as well. Applying Proposition 2.12 to the reverse of $F' - \underline{u}$ we get that $F' - \underline{u}$ is Turánable too. Thus, due to Theorem 2.4 we can embed $F' - \underline{u}$ and $F' - \overline{u}$ into any canonical ordering of K_{f+1}. We will use these embeddings to find embeddings of F' into each $\star$-canonical ordering of K_{f+2}.

For the smaller orderings of K_{f+2}, we first embed $F' - \underline{u}$ into the canonical part $K_{f+2} - x$, and then embed $\underline{u}$ as the special vertex x. In this way, the edge $\underline{uv}$ is embedded onto an edge of the form xv_i; therefore, by definition of the smaller orderings, the edge $\underline{uv}$ is embedded onto is smaller than all edges in the embedding of $F' - \underline{u}$. This gives rise to a copy of F'.

For the larger orderings of K_{f+2} the proof is analogous, by embedding $F' - \overline{u}$ into the canonical part $K_{f+2} - x$ and then embedding $\overline{u}$ onto x.

For the middle increasing $\star$-canonical orderings of K_{f+2}, we now split into subcases depending on its canonical part.

[5]We highlight that there might be more than one minimal (resp. maximal) vertex, as there might be more than one embedding of F into a min (resp. max) ordering. For example, in a monotone path $u_1u_2u_3u_4$, we have that u_1 and u_2 can play the role of v_1 in a min ordering.

If the canonical part is a min ordering, since $\underline{v}$ is a minimal vertex in F, there is an embedding of F into $K_{f+2}[\{v_1, \dots, v_f\}]$ such that $\underline{v} \mapsto v_1$. Let $i \in [f] \setminus \{1\}$ be such that $\overline{v} \mapsto v_i$ in that embedding. Observe that, for every edge w_1w_2 in F such that $w_1 \mapsto v_j$ and $w_2 \mapsto v_k$ for a pair of indices $j, k \in [f] \setminus \{i\}$, we have

$$v_jv_k < v_iv_{f+1} \tag{2.9}$$

in the edge-ordering of K_{f+2}. To see this, observe that since $\overline{v}$ is maximal and not isolated in F, $\overline{v}$ must be contained in the maximal edge of F, and hence, the embedding of the maximal edge must be of the form v_iv_ℓ for some $\ell \in [f] \setminus \{i\}$. Thus, if (2.9) does not hold for some edge w_1w_2 in F, then

$$v_jv_k > v_iv_{f+1} > v_iv_\ell \,,$$

where the last inequality holds since the canonical part is a min ordering and $\ell < f+1$. However, this is a contradiction since the maximal edge in F is embedded onto v_iv_ℓ. Now we extend this embedding to an embedding of $F' - \underline{u}$ by taking $\overline{u} \mapsto v_{f+1}$. Indeed, the edge $\overline{uv}$ is embedded onto v_iv_{f+1} which, due to (2.9), is larger than any edge in our copy of F, implying a copy of $F' - \underline{u}$ in K_{f+1}. Finally, extend the embedding further by taking $\underline{u} \mapsto x$. Observe that the edge $\underline{uv}$ is embedded in this way onto the edge xv_1. Moreover, by Remark 2.8, $xv_1 < v_1v_2$ and v_1v_2 is the smallest edge in the canonical part by Definition 2.2. Therefore, the edge $\underline{uv}$ is embedded onto is smaller than all other edges used. Thus, we find a copy of F'.

If the canonical part is an inverse min ordering, we embed $F' - \overline{u}$ into the canonical part and then take $\overline{u} \mapsto x$. Let v_i be the vertex $\overline{v}$ is embedded onto (where $i \in [f]$). Note that

$$xv_i > \max\{v_iv_j \colon j \in [f] \setminus \{i\}\} \,. \tag{2.10}$$

Indeed, using the labelings given by Definitions 2.2 and 2.7 we have $L_3(xv_i) = 2fi > (2f+1)i - j = L_3(v_iv_j)$ for every $i < j \leq f$ and $L_3(xv_i) = 2fi > (2f+1)j - i = L_3(v_iv_j)$ for every $1 \leq j < i$. Thus, (2.10) implies that the edge xv_i that $\overline{uv}$ is embedded onto is larger than any of the edges in our copy of $F' - \overline{u}$ that contain $\overline{v}$. Since $\overline{v}$ is a maximal non-isolated vertex in F, $\overline{v}$ is contained in the maximal edge of F. The maximal edge of F is also the maximal edge of $F' - \overline{u}$ and therefore, the edge xv_i that $\overline{uv}$ is embedded onto is larger any of the edges in our copy of $F' - \overline{u}$. As before, this yields a copy of F' in K_{f+2}.

Finally, if the canonical part is a max ordering or an inverse max ordering we argue as before, but for the reverse graph $\overleftarrow{F'}$. More precisely, note first that for $\overleftarrow{F}$ the vertices $\underline{v}$ and $\overline{v}$ are maximal and minimal respectively. Moreover, if F'' is constructed from $\overleftarrow{F}$ by adding two new vertices $\underline{w}, \overline{w}$ and the edges $\underline{wv}$ and $\overline{wv}$ such that $\underline{wv}$ is larger than all other edges and $\overline{wv}$ is smaller than all other edges, then F'' is precisely the reverse of F'. By the argument above, a middle increasing ordering of K_{f+2} whose canonical part is a min or an inverse min ordering contains a copy of F''. Hence, the reverse of that ordering contains a copy of F'. We conclude by noticing that reverse of a middle increasing ordering whose canonical part is the min (resp. inverse min) ordering is the middle increasing ordering whose canonical part is the max (resp. inverse max) ordering. □

Proposition 2.11 implies that no edge-ordering of K_4^- is tileable. In contrast, the following corollary of Proposition 2.14 asserts that there are connected tileable edge-ordered graphs containing K_4^-. Recall D_4 is a Turánable edge-ordering of K_4^-; further notice D_4 has unique minimal and maximal vertices, and they are distinct.

Corollary 2.15. *For every even $n \geq 6$ there is a connected n-vertex tileable edge-ordered graph F_n with $K_4^- \subseteq F_n$.*

Proof. We proceed by induction on $n \geq 6$. For $n = 6$, apply Proposition 2.14 with $F := D_4$, and let F_6 be the resulting edge-ordered graph. Since F_6 is tileable and $K_4^- \subseteq F_6$, we establish the base case. Notice that one of the new vertices in F_6 is minimal, the other is a maximal vertex. Similarly, suppose that F_n is a connected n-vertex tileable edge-ordered graph with distinct minimal and maximal vertices so that $K_4^- \subseteq F_n$. Then we apply Proposition 2.14 with F_n playing the role of F and let F_{n+2} be the output of this proposition. Notice that F_{n+2} is a connected $(n+2)$-vertex tileable edge-ordered graph with $K_4^- \subseteq F_n \subseteq F_{n+2}$. Moreover, F_{n+2} will contain distinct minimal and maximal vertices (the two new vertices). □

In Proposition 2.14 we obtain a tileable edge-ordered graph from a Turánable edge-ordered graph by adding *two* pendant edges. The following proposition shows that adding only one such pendant edge is, in general, not enough to create a tileable edge-ordered graph. Recall we write $u_1, \ldots, u_n$ for the vertices of D_n where u_1 and u_n are the unique minimal and maximal vertices in D_n, respectively.

Proposition 2.16. *For $n \geq 4$, let D_n^+ be the edge-ordered graph obtained from D_n by adding a new vertex w and the edge $u_n w$, larger than all the edges in D_n. Let D_n^- be the edge-ordered graph obtained from D_n by adding a new vertex u and the edge $u_1 u$, smaller than all the edges in D_n. Then neither D_n^+ nor D_n^- are tileable.*

Proof. We only consider D_n^+ as the argument for D_n^- is analogous. Suppose for a contradiction there is an embedding of D_n^+ into a smaller decreasing ordering of K_{n+1} whose canonical part is a min ordering. First, since $u_1u_n < u_iu_n$ for $1 < i < n$ and u_nw is the largest edge in D_n^+, u_1 is the only vertex in D_n^+ such that all edges incident to it are smaller than all other edges. Note that this means we must have that $u_1 \mapsto x$. Recall that $u_1u_2 < \cdots < u_1u_n$ in D_n and that, since the $\star$-canonical ordering of K_{n+1} is smaller decreasing, $v_1x > \cdots > v_nx$. Thus, given $1 < i < j \leq n$,

$$\text{if } u_i \mapsto v_k \text{ and } u_j \mapsto v_\ell \text{ then } \ell < k.$$

In particular, if we take $i, j, k \in [n]$ such that

$$u_n \mapsto v_i\,, \qquad u_3 \mapsto v_j\,, \qquad \text{and} \qquad u_2 \mapsto v_k,$$

then $i < j < k$. However, this is a contradiction, because while $u_nu_2 < u_nu_3$ in D_n^+, we have $v_iv_k > v_iv_j$ in K_{n+1}. □

Since the operations described in Propositions 2.13 and 2.14 add pendant edges, the case in which the underlying graph is a cycle is not directly covered by them. In the following two propositions we study the tileability of *monotone cycles*. We say that an edge-ordered cycle C_n with $V(C_n) = \{u_1, \ldots, u_n\}$ is *monotone* if the edges are ordered as $u_1u_2 < u_2u_3 < \cdots < u_{n-1}u_n < u_nu_1$.

Proposition 2.17. *Monotone cycles of odd length are tileable.*

Proof. It suffices to find a spanning monotone cycle in every $\star$-canonical ordering of K_{n+1} where n is even. For this, we show that every canonical ordering of K_n contains an embedding of the monotone spanning path which can be extended to the special vertex x on both ends so that the resulting cycle is monotone.

We now define four paths in the canonical orderings with vertex set $\{v_1, \ldots, v_n\}$, and state in which canonical orderings they are in fact monotone paths.

- *Ordinary*: $v_1v_2v_3 \ldots v_n$ is monotone in all four canonical orderings.
- *Small*: $v_2v_3 \ldots v_nv_1$ is monotone in the inverse max ordering.
- *Big*: $v_nv_1v_2 \ldots v_{n-1}$ is monotone in the inverse min ordering.
- *Jumpy*: $v_{n/2+1}v_1v_{n/2+2}v_2 \cdots v_nv_{n/2}$ is monotone in the min ordering and max ordering.

For each $\star$-canonical ordering of K_{n+1}, we now show how to extend one of the previous monotone paths into a spanning monotone cycle using the special vertex x.

For all larger/smaller decreasing orderings, we simply extend the ordinary path by adding the special vertex x 'between' v_n and v_1. The resulting cycle is monotone since, by Definition 2.7,

- for larger decreasing orderings $v_1v_2 < \ldots < v_{n-1}v_n < v_nx < xv_1$;

- for smaller decreasing orderings $v_n x < xv_1 < v_1 v_2 < \ldots < v_{n-1} v_n$.

The remaining $\star$-canonical orderings are all increasing. We split the analysis into cases depending on their canonical part.

Suppose first that the canonical part is a min or a max ordering. For the middle increasing ordering observe Remark 2.8 implies that for the min and max orderings, xv_1 and xv_n are the smallest and largest edges respectively. Then, we simply take the ordinary path and add the special vertex between v_n and v_1 to get a monotone cycle

$$xv_1 < v_1 v_2 < \ldots < v_{n-1} v_n < v_n x \, .$$

If the ordering is smaller or larger increasing we extend a jumpy path by adding the special vertex x between $v_{n/2}$ and $v_{n/2+1}$ as we have $xv_{n/2} < xv_{n/2+1}$ for all increasing orderings. Observe that the resulting cycle is monotone, since

- for larger increasing orderings $v_{n/2+1} v_1 < \ldots < v_n v_{n/2} < v_{n/2} x < xv_{n/2+1}$;
- for smaller increasing orderings $v_{n/2} x < xv_{n/2+1} < v_{n/2+1} v_1 < \ldots < v_n v_{n/2}$.

Suppose now that the canonical part is an inverse min ordering. We extend the big path by adding the special vertex between v_{n-1} and v_n. By Definition 2.7 and Remark 2.8, observe that for the larger and middle increasing orderings,

$$v_n v_1 < \ldots < v_{n-2} v_{n-1} < v_{n-1} x < xv_n \, ,$$

while for the smaller increasing ordering,

$$v_{n-1} x < xv_n < v_n v_1 < \ldots < v_{n-2} v_{n-1} \, .$$

Finally, suppose the canonical part is an inverse max ordering; we extend the small path by adding the special vertex between v_1 and v_2. Indeed, by Definition 2.7 and Remark 2.8, observe that for the smaller and middle increasing orderings,

$$v_1 x < xv_2 < v_2 v_3 < \ldots < v_n v_1 \, ,$$

while for the larger increasing ordering,

$$v_2 v_3 < \ldots < v_n v_1 < v_1 x < xv_2 \, . \qquad \square$$

In stark contrast to Proposition 2.17, the next result states that monotone cycles of even length are not Turánable, let alone tileable.

Proposition 2.18. *Monotone cycles of even length are not Turánable.*

Proof. By Theorem 2.4, it suffices to show that there is no spanning monotone cycle in the min canonical ordering of K_n for n even. We will proceed by induction on n.

Before this, we first show that in the min ordering of K_n,

$$\text{if } v_i v_j < v_j v_k \text{ then } i < k. \tag{2.11}$$

Indeed, suppose $k < i$. Using the standard labeling of Definition 2.2, we have that if $j < k$ then $2nj + i - 1 = L_1(v_i v_j) < L_1(v_j v_k) = 2nj + k - 1$, which is a contradiction. If $k < j < i$, then $2nj + i - 1 = L_1(v_i v_j) < L_1(v_j v_k) = 2nk + j - 1$, which implies that $2n(j-k) < j - i$; this is a contradiction, since $k < j$ while $j < i$. Finally, if $k < i < j$, then $2ni + j - 1 = L_1(v_i v_j) < L_1(v_j v_k) = 2nk + j - 1$, which again is a contradiction.

Let C_4^{mon} be a monotone cycle of length four with vertices u_1, u_2, u_3, u_4 and edges ordered as $u_1 u_2 < u_2 u_3 < u_3 u_4 < u_4 u_1$. Suppose there is an embedding of C_4^{mon} into the min ordering of K_4 and let $i, k \in [4]$ be such that

$$u_1 \mapsto v_i \qquad \text{and} \qquad u_3 \mapsto v_k \, .$$

Since $u_1u_2 < u_2u_3$ and due to (2.11), we have $i < k$, but similarly, since $u_3u_4 < u_4u_1$, we have $k < i$, a contradiction.

Now suppose that the min ordering of K_n does not contain a spanning monotone cycle C_n^{mon} for some even $n \geq 4$. Let $\{u_1, \dots, u_{n+2}\}$ be the vertex set of a monotone cycle C_{n+2}^{mon}, with edges ordered as $u_1u_2 < \dots < u_{n+1}u_{n+2} < u_{n+2}u_1$. Suppose for contradiction there is an embedding

$$\varphi \colon V(C_{n+2}^{\mathrm{mon}}) \longrightarrow V(K_{n+2})$$

of C_{n+2}^{mon} into the min ordering of K_{n+2}. First, we shall check that for any four vertices v_i, v_j, v_k, v_ℓ in the min ordering of K_{n+2},

$$\text{if } v_iv_j < v_jv_k < v_kv_\ell \text{ we have that } v_iv_\ell < v_kv_\ell\,. \tag{2.12}$$

Indeed, since $v_iv_j < v_jv_k$, we have that (2.11) yields $i < k$. If we suppose $v_kv_\ell < v_iv_\ell$, then again (2.11) implies that $k < i$, which is a contradiction, and therefore (2.12) follows.

Due to (2.12), and since $\varphi(u_1)\varphi(u_2) < \varphi(u_2)\varphi(u_3) < \varphi(u_3)\varphi(u_4)$ in K_{n+2}, we have $\varphi(u_1)\varphi(u_4) < \varphi(u_3)\varphi(u_4) < \varphi(u_4)\varphi(u_5)$. Hence, we have that

$$\varphi(u_1)\varphi(u_4) < \varphi(u_4)\varphi(u_5) < \varphi(u_5)\varphi(u_6) < \dots < \varphi(u_{n+1})\varphi(u_{n+2}) < \varphi(u_{n+2})\varphi(u_1)\,,$$

which is a copy of a monotone cycle of length n embedded into the edge ordered graph induced by the vertices $V(K_{n+2}) \setminus \{\varphi(u_2), \varphi(u_3)\}$. But this is a contradiction to our induction hypothesis since, due to Fact 2.5, $V(K_{n+2}) \setminus \{\varphi(u_2), \varphi(u_3)\}$ induces a min ordering of K_n. □

2.3. **Proof of Theorem 2.6.** First we prove the following lemma that provides an alternative characterization of tileable edge-ordered graphs.

Lemma 2.19. *An edge-ordered graph F is tileable if and only if there exists an $n \in \mathbb{N}$ such that the following holds. Every edge-ordering of K_n such that $K_n - x$ is canonical for some vertex $x \in V(K_n)$ contains a copy of F that covers x.*

Proof. For the 'forwards direction', suppose that there is no $n \in \mathbb{N}$ satisfying the property described in the lemma. That is, for every $n \in \mathbb{N}$ there is an edge-ordering of the complete graph K_n such that $K_n - x$ is canonically edge-ordered for some vertex $x \in V(K_n)$ and x is not contained in any copy of F. In particular, none of these edge-ordered complete graphs contain an F-tiling covering x, and so F is not tileable.

For the 'backwards direction', let $n \in \mathbb{N}$ be as in the statement of the lemma and set $f := |V(F)|$. We shall prove that F is tileable, that is, there exists a $t \in \mathbb{N}$ such that every edge-ordering of K_t contains a perfect F-tiling. Note first that the property of n guarantees that every canonical edge-ordering of K_n contains a copy of F. In particular, Fact 2.5 implies that for every $\ell \in \mathbb{N}$, every canonical edge-ordering of $K_{\ell f}$ contains a perfect F-tiling. Further, given $k \geq n$ where k is divisible by f, if K_k is such that $K_k - x$ is canonically edge-ordered for some vertex $x \in V(K_k)$, then K_k contains a perfect F-tiling. Indeed, by the property of n, K_k contains a copy F' of F with $x \in V(F')$; hence, as $K_k \setminus V(F')$ is canonically edge-ordered, the discussion above implies that $K_k \setminus V(F')$, and thus K_k, contains a perfect F-tiling.

Pick $k \geq n$ such that k is divisible by f and let $m \in \mathbb{N}$ be the output of Proposition 2.1 on input $k-1$. Fix $t := (m-1)k$ and let $K := K_t$ be arbitrarily edge-ordered. Apply Proposition 2.1 iteratively $m-1$ times to find vertex-disjoint copies of K_{k-1} in K, each of them canonically edge-ordered. Let $K_{k-1}^{(1)}, \dots, K_{k-1}^{(m-1)} \subseteq K$ be these copies and observe that exactly $m-1$ vertices remain uncovered in K. That is, there are vertices $x_1, \dots, x_{m-1}$ such that $V(K) = \bigcup_{i \in [m-1]} V(K_{k-1}^{(i)}) \cup \{x_i\}$. By the discussion above, for every $i \in [m-1]$, $K[V(K_{k-1}^{(i)}) \cup \{x_i\}]$ contains a perfect F-tiling and hence, K contains a perfect F-tiling as well, as required. □

In the proof of Theorem 2.6 we deal with canonical orderings of K_n with vertex set $\{v_1, \dots, v_n\}$. Let $U \subseteq V(K_n)$ be a subset of size $k \leq n$ such that $U = \{v_{i_1}, \dots, v_{i_k}\}$ where $j < k$ implies $i_i < i_k$. Whenever we say that we *relabel the vertices of* U, we mean that we will denote v_{i_j} simply as v_j (and we will restrict our attention to this subset of the original vertex set).

Proof of Theorem 2.6. Suppose F is tileable; by definition there is some $n \in \mathbb{N}$ so that in any $\star$-canonical ordering of K_{n+1} there is a perfect F-tiling. In such a perfect F-tiling there is a copy F' of F that contains the special vertex x. Fact 2.9 implies that $K_{n+1}[V(F')]$ is $\star$-canonically edge-ordered with the same type as K_{n+1}. Thus, this implies every $\star$-canonical ordering of K_f contains a copy of F.

For the other direction, suppose every $\star$-canonical ordering of K_f contains a copy of F. Our aim is to show that F is tileable. By Lemma 2.19, it suffices to prove that there is an $n \in \mathbb{N}$ such that every edge-ordering of K_{n+1} for which $K_{n+1} - x$ is canonically ordered for some vertex $x \in V(K_{n+1})$, contains a copy of F that covers x.

The cases $f = 2, 3$ are trivial, so we may assume $f \geq 4$. Let $n \in \mathbb{N}$ be sufficiently large compared to $f \geq 4$ and where $\sqrt{n-1} \in \mathbb{N}$. Let $\{x, v_1, \dots, v_n\}$ be the vertices of an edge-ordered complete graph K_{n+1}, such that $K_{n+1} - x$ is canonically ordered. Our goal is to find a subgraph $K_f \subseteq K_{n+1}$ containing x such that K_f is $\star$-canonically edge-ordered. Indeed, by our assumption this K_f contains a copy of F, and so K_{n+1} contains a copy of F that covers x, as desired.

Observe that an application of the Erdős–Szekeres Theorem [9] to the sequence of edges $\{xv_i\}_{i \in [n]}$ yields a monotone subsequence. More precisely, there is a set $I \subseteq [n]$ of size at least $\sqrt{n-1} + 1$ such that the sequence $\{xv_i\}_{i \in I}$ is monotone. Further, let $V_I := \{v_i\}_{i \in I}$ and consider the 3-coloring $c : E(K_{n+1}[V_I]) \to \{B, M, S\}$ of the edges of $K_{n+1}[V_I]$ defined as follows: for $i, j \in I$ with $i < j$, let

$$c(v_iv_j) := \begin{cases} B & \text{if } xv_i, xv_j > v_iv_j\,, \\ M & \text{if } xv_i < v_iv_j < xv_j \text{ or } xv_j < v_iv_j < xv_i\,, \text{and} \\ S & \text{if } xv_i, xv_j < v_iv_j\,. \end{cases}$$

As n is sufficiently large, Ramsey's Theorem implies that there is a monochromatic clique $\widetilde{K}$ on $\ell := f^2 - 4f + 5$ vertices. Relabeling the vertices of $V(\widetilde{K})$ we take $V(\widetilde{K}) = \{v_1, \dots, v_\ell\}$ and thus we have

(1) $\widetilde{K}$ is canonically ordered;
(2) $\{xv_i\}_{i \in [\ell]}$ is a monotone sequence;
(3) exactly one of the following holds:
   (a) $xv_i, xv_j > v_iv_j$ for every $1 \leq i < j \leq \ell$,
   (b) $xv_i, xv_j < v_iv_j$ for every $1 \leq i < j \leq \ell$, or
   (c) $xv_i < v_iv_j < xv_j$ or $xv_j < v_iv_j < xv_i$ for every $1 \leq i < j \leq \ell$.

We shall prove that $K_{n+1}[V(\widetilde{K}) \cup \{x\}]$ contains a $\star$-canonically edge-ordered copy of K_f containing the vertex x, as desired. We split the rest of the proof into cases depending on whether the sequence $\{xv_i\}_{i \in [\ell]}$ is increasing or decreasing, and depending on which of (3a), (3b), and (3c) holds.

Case (1) The sequence $\{xv_i\}_{i \in [\ell]}$ is decreasing and (3a) holds.

Note that $xv_1 > xv_2 > \cdots > xv_\ell > \max\{v_iv_\ell \colon 1 \leq i < \ell\} = \max\{v_iv_j \colon 1 \leq i < j \leq \ell\}$, where the last equality follows as in any canonical edge-ordering of K_ℓ the largest edge is incident to v_ℓ. Thus, $K_{n+1}[V(\widetilde{K}) \cup \{x\}]$ is a $\star$-canonically edge-ordered copy of $K_{\ell+1}$ with special vertex x, and with larger decreasing ordering.

Case *(2)* The sequence $\{xv_i\}_{i\in[\ell]}$ is decreasing and (3b) holds.

Note that $xv_\ell < \cdots < xv_1 < \min\{v_1v_i\colon 1 < i \leq \ell\} = \min\{v_iv_j\colon 1 \leq i < j \leq \ell\}$, where the last equality follows as in any cannonical edge-ordering of K_ℓ the smallest edge is incident to v_1. Thus, $K_{n+1}[V(\widetilde{K}) \cup \{x\}]$ is a $\star$-canonically edge-ordered copy of $K_{\ell+1}$ with special vertex x, and with smaller decreasing ordering.

Case *(3)* The sequence $\{xv_i\}_{i\in[\ell]}$ is decreasing and (3c) holds.

As $xv_1 > xv_2 > \cdots > xv_\ell$, (3c) implies that $xv_1 > v_1v_2 > xv_2$ and also $xv_2 > v_2v_j > xv_j$ for all $3 \leq j \leq \ell$. Thus, $v_1v_2 > \max\{v_2v_i\colon 2 < i \leq \ell\}$. Note though, however $\widetilde{K}$ is canonically ordered, we must have that $\max\{v_2v_i\colon 2 < i \leq \ell\} > v_1v_2$. Since this is a contradiction, this case cannot happen.

Case *(4)* The sequence $\{xv_i\}_{i\in[\ell]}$ is increasing and (3c) holds.

In this case notice that for all $k \in [\ell-1]$ we have

$$xv_k < v_kv_{k+1} < xv_{k+1}. \tag{2.13}$$

Furthermore,

$$\begin{aligned} \max\{v_iv_k\colon 1 \leq i < k\} <& xv_k \text{ for all } 2 \leq k \leq \ell \quad \text{and} \\ & xv_k < \min\{v_kv_i\colon k < i \leq \ell\} \text{ for all } k \in [\ell-1]. \end{aligned} \tag{2.14}$$

When $\widetilde{K}$ is an inverse min (resp. inverse max) ordering, (2.13) and (2.14) imply that $\{x, v_1, \ldots, v_\ell\}$ induces a canonical ordering of the same type as $\widetilde{K}$, with x as the last (resp. first) vertex.

When $\widetilde{K}$ is a min ordering, we have

$$v_iv_\ell \;<\; v_{i+1}v_{i+2} \overset{(2.13)}{<} xv_{i+2} \overset{(2.14)}{<} v_{i+2}v_{i+4}\,,$$

where the first inequality follows by Remark 2.3. Since $\ell = f^2-4f+5 \geq 2f-3$ for $f \geq 4$, restricting to the vertices of odd index in $\widetilde{K}$, we obtain from Remark 2.8 that $K_{n+1}[\{x, v_1, v_3, \ldots, v_{2f-3}\}]$ is a $\star$-canonically edge-ordered copy of K_f with special vertex x, and with middle increasing ordering.

For the max ordering, we use an analogous argument: (2.14) implies $v_iv_{i+2} < xv_{i+2} < v_{i+2}v_{i+3} < v_1v_{i+4}$. Using again Remark 2.8 we have that $K_{n+1}[\{x, v_1, v_3, \ldots, v_{2f-3}\}]$ is a $\star$-canonically edge-ordered copy of K_f with special vertex x, and with middle increasing ordering.

Case *(5)* The sequence $\{xv_i\}_{i\in[\ell]}$ is increasing and (3a) holds.

We separate the proof of this case into three claims.

Claim 2.20. *If $\widetilde{K}$ is a max or an inverse max ordering then $K_{n+1}[V(\widetilde{K}) \cup \{x\}]$ contains a $\star$-canonically edge-ordered copy of K_f with larger increasing ordering and special vertex x.*

Proof of the claim: For these canonical orderings we have $v_1v_\ell > \max\{v_iv_j\colon 1 \leq j \leq \ell-1\}$. Then, due to (3a), we have

$$xv_\ell > \cdots > xv_2 > xv_1 > v_1v_\ell > \max\{v_iv_j\colon 1 \leq i < j \leq \ell-1\}\,,$$

and therefore $\{x, v_1, \ldots, v_{\ell-1}\}$ induces a larger increasing ordering. ■

When $\widetilde{K}$ is a min or an inverse min ordering we will use the following claim.

Claim 2.21. *Suppose $\widetilde{K}$ is a min or an inverse min ordering. Either $K_{n+1}[V(\widetilde{K})\cup\{x\}]$ contains a larger increasing $\star$-canonical ordering of K_f containing x or the following statement holds. There is a set $U_{f-3} \subseteq V(\widetilde{K})$ such that, after relabeling the vertices, we have $U_{f-3} := \{v_1, \ldots, v_{f-1}\}$ and, for all $i < f-2$,*

$$\max\{v_iv_j\colon i < j \leq f-1\} < xv_i < \min\{v_jv_k\colon i < j < k \leq f-1\}\,. \tag{2.15}$$

Proof of the claim: Suppose $\widetilde{K}$ is a min or an inverse min ordering and $K_{n+1}[V(\widetilde{K})\cup\{x\}]$ does not contain a larger increasing $\star$-canonical ordering of K_f containing x. For each $0 \le r \le f-3$, define $\ell_r := \ell - r(f-2)$; so $\ell_0 = \ell$ and

$$\ell_{f-3} = \ell - (f-3)(f-2) = (f^2-4f+5) - (f-3)(f-2) = f-1\,.$$

To prove the claim we proceed iteratively as follows. Suppose for some $0 \le r < f-3$ there is a set of vertices $U_r := \{v_1, \dots, v_{\ell_r}\}$ satisfying

$$\max\{v_iv_j\colon i<j\le \ell_r\} < xv_i < \min\{v_jv_k\colon i<j<k\le \ell_r\} \text{ for all } i \le r. \tag{2.16}$$

We shall find a set $U_{r+1} \subseteq U_r$ such that, after relabeling, we have $U_{r+1} := \{v_1, \dots, v_{\ell_{r+1}}\}$ and where (2.16) holds for $r+1$ instead of r. To start the iteration take $r=0$ and let $U_0 := V(\widetilde{K})$.

If $xv_{r+1} > \max\{v_jv_k\colon r<j<k<r+f\}$, then, since $\{xv_j\}_{j\in[\ell]}$ is increasing, we have

$$xv_{r+f-1} > xv_{r+f-2} > \dots > xv_{r+1} > \max\{v_jv_k\colon r<j<k<r+f\}\,.$$

Thus, $\{x, v_{r+1}, \dots, v_{r+f-1}\}$ induces a larger increasing $\star$-canonical ordering of K_f contradicting our initial supposition. So we may assume that $xv_{r+1} < \max\{v_jv_k\colon r<j<k<r+f\}$ and conclude

$$\begin{aligned}\max\{v_{r+1}v_i\colon r+1<i\le \ell_r\} \overset{(3a)}{<} xv_{r+1} \;&<\; \max\{v_jv_k\colon r<j<k<r+f\}\\ &<\; \min\{v_jv_k\colon r+f\le j<k\le \ell_r\}\,.\end{aligned} \tag{2.17}$$

The last inequality follows from the fact that $\widetilde{K}$ is min or inverse min ordered and by recalling Remark 2.3.

Delete the vertices $v_{r+2}, \dots, v_{r+f-1}$, relabel the remaining vertices, and let $U_{r+1} := \{v_1, \dots, v_{\ell_{r+1}}\}$ be the set of vertices after the deletion and the relabeling. We shall prove that U_{r+1} satisfies (2.16) for $r+1$ instead of r. First, observe that for $i \le r+1$, v_i is not deleted and keeps the same label as in U_r. Moreover, since we only delete vertices, the sets from which we take the maximum and minimum in (2.16) are now smaller, and thus, for $i \le r$, (2.16) becomes in fact less restrictive after the deletion and relabeling. Therefore, the inequalities in (2.16) still hold for $i \le r$ with ℓ_{r+1} instead of ℓ_r. We still need to prove that they hold for $i = r+1$. For that, note that vertex v_{r+f} is relabeled as v_{r+2} in U_{r+1} and therefore (2.17) implies

$$\max\{v_{r+1}v_i\colon r+1<i\le \ell_{r+1}\} < xv_{r+1} < \min\{v_jv_k\colon r+2\le j<k\le \ell_{r+1}\}\,,$$

in U_{r+1}. That is, the inequalities in (2.16) hold for $i = r+1$ in U_{r+1} and with ℓ_{r+1} instead of ℓ_r. Hence, (2.16) holds for $r+1$ instead of r.

Since $\ell_{f-3} = f-1$, after $f-3$ steps we obtain $U_{f-3} = \{v_1, \dots, v_{f-1}\}$ satisfying (2.15) for every $i < f-2$. ■

We use Claim 2.21 to prove the following claim finishing the proof of this case.

Claim 2.22. *Suppose $\widetilde{K}$ is a min or an inverse min ordering. Either $K_{n+1}[V(\widetilde{K})\cup\{x\}]$ contains a larger increasing $\star$-canonical ordering of K_f containing x or the following two statements hold.*

- *If $\widetilde{K}$ is a min canonical ordering then $K_{n+1}[V(\widetilde{K})\cup\{x\}]$ contains a min canonically ordered copy of K_f containing x.*
- *If $\widetilde{K}$ is an inverse min canonical ordering then $K_{n+1}[V(\widetilde{K})\cup\{x\}]$ contains a middle increasing $\star$-canonical ordering of K_f with special vertex x.*

Proof of the claim: Suppose $\widetilde{K}$ is a min or an inverse min ordering and $K_{n+1}[V(\widetilde{K})\cup\{x\}]$ does not contain a larger increasing $\star$-canonical ordering of K_f containing x. Apply Claim 2.21 to obtain a set U such that after relabeling the vertices we have $U := \{v_1, \dots, v_{f-1}\}$ satisfying (2.15) for every $i < f-2$.

Since the sequence $\{xv_i\}_{i\in[\ell]}$ is increasing and because of (3a) we deduce

$$v_{f-2}v_{f-1} < v_{f-2}x < v_{f-1}x. \tag{2.18}$$

If $\widetilde{K}$ is a min canonical ordering then (2.15) becomes $v_iv_{f-1} < xv_i < v_{i+1}v_{i+2}$ for $i < f-2$. Then, using (2.18) it is easy to check that $U \cup \{x\}$ induces a min canonical ordering, with x playing the role of the last vertex v_f. If $\widetilde{K}$ is an inverse min canonical ordering, then (2.15) becomes $v_iv_{i+1} < xv_i < v_{i+1}v_{f-1}$ for every $i < f-2$. Then, we obtain from (2.18) and Remark 2.8 that $U \cup \{x\}$ induces a middle increasing ordering with special vertex x. ■

Case (6) The sequence $\{xv_i\}_{i\in[\ell]}$ is increasing and (3b) holds.

For this case we reverse the edge-ordering of $K_{n+1}[V(\widetilde{K}) \cup \{x\}]$ and the ordering of the vertices in the canonical part. More precisely, let $\overleftarrow{K} := \overleftarrow{K}_{n+1}[V(\widetilde{K}) \cup \{x\}]$ be the reverse of $K_{n+1}[V(\widetilde{K}) \cup \{x\}]$ and let $V(\overleftarrow{K}) \setminus \{x\}$ be reordered as $V(\overleftarrow{K}) \setminus \{x\} = \{v'_1, \dots, v'_\ell\}$ where $v'_i := v_{\ell-i+1}$. Then

(*1*) $\overleftarrow{K}[V(\widetilde{K})]$ is canonically ordered,
(*2*) $\{xv'_i\}_{i\in[\ell]}$ is increasing, and
(*3*) (3a) holds for $\overleftarrow{K}$.

Indeed, for *(1)* notice that the reverse of a canonical ordering is canonical after reversing the ordering of the vertices. For *(2)* observe that we reverse the ordering of the vertices and edges, so the sequence is still increasing. Finally, *(3)* is easy to deduce after noticing that (3a) and (3b) only depend on the ordering of the edges and not on the ordering of the vertices.

Observe that conditions *(1)*–*(3)* are the same conditions we have for *Case (5)*. Thus, to address our current case, we apply Claims 2.20 and 2.22 to the edge-ordered graph $\overleftarrow{K}$.

More precisely, when $\widetilde{K}$ is a min ordering or an inverse min ordering, then $\overleftarrow{K}$ is a max or an inverse max ordering. Therefore, Claim 2.20 implies that $\overleftarrow{K}$ contains a $\star$-canonically edge-ordered copy of K_f with larger increasing ordering and special vertex x. Hence, $K_{n+1}[V(\widetilde{K}) \cup \{x\}]$ contains a $\star$-canonically edge-ordered copy of K_f with smaller increasing ordering and special vertex x.

By an analogous argument but using Claim 2.22 instead of Claim 2.20, we have that if $\widetilde{K}$ is a max ordering or an inverse max ordering then $K_{n+1}[V(\widetilde{K}) \cup \{x\}]$ contains a $\star$-canonical ordering copy of K_f containing x. Moreover, this copy of K_f is either a smaller increasing ordering, a max canonical ordering, or a middle increasing ordering. □

3. Universally tileable graphs

We begin this section with the proof of Theorem 1.5.

Proof of Theorem 1.5. To prove the statement we will show that *(a)* implies *(b)*, *(b)* implies *(c)* and *(c)* implies *(a)*. If an edge-ordered graph is tileable then by definition it is Turánable. Thus, *(a)* immediately implies *(b)*. Theorem 2.18 from [11] already proves that *(b)* implies *(c)*. It therefore remains to show that *(c)* implies *(a)*.

First assume that H is a K_3 together with a (possibly empty) collection of isolated vertices. Note that all edge-orderings of H are isomorphic, so every edge-ordering of $K_{|H|}$ contains a spanning copy of $H^{\leq}$, for every edge-ordering $\leq$. Thus H is universally tileable.

Now, suppose H is a path on three edges. There are three types of edge-ordering of H; 123, 132, and 213. The latter two are contained in any edge-ordering of C_4 and so are tileable. The former is just $P_3^{\leq}$, so is tileable by Theorem 1.1. Thus, H is universally tileable. Note that adding isolated vertices to a tileable edge-ordered graph results in another tileable edge-ordered graph. Therefore, every path on three edges together with a (possibly empty) collection of isolated vertices forms a universally tileable graph.

Finally, assume that H is a star forest and $H^{\leq}$ is any edge-ordering of H. Let $h := |H|$. We now check that we can find a copy of $H^{\leq}$ in any $\star$-canonically edge-ordered K_h. As usual we write $\{x, v_1, \ldots, v_{h-1}\}$ for the vertices of a $\star$-canonically edge-ordered K_h, where x is the special vertex.

Given any vertex v in $H^{\leq}$, $H^{\leq} - v$ is a star forest and so is Turánable by [11, Theorem 2.18]; thus, by Theorem 2.4, any canonical ordering of K_{h-1} contains a copy of $H^{\leq} - v$.

Consider any smaller increasing/decreasing $\star$-canonical ordering of K_h. Let uw be the smallest edge in $H^{\leq}$ where u is a leaf of $H^{\leq}$. By the remark in the previous paragraph, our edge-ordered K_h contains a copy of $H^{\leq} - u$ that does not contain x. By definition of a smaller increasing/decreasing $\star$-canonical ordering, we can now add x to this copy of $H^{\leq} - u$ to obtain a copy of $H^{\leq}$ in our edge-ordered K_h. For a larger increasing/decreasing $\star$-canonical ordering of K_h one can argue analogously, but take uw to be the largest edge in $H^{\leq}$, instead of the smallest.

In a middle increasing $\star$-canonical ordering of K_h we need to explicitly define how to embed $H^{\leq} - v$ into the canonical orderings of K_{h-1}. Let $\{K_{1,t_i}\}_{1 \leq i \leq k}$ be the collection of k stars that form the components of H and let $C \subseteq V(H)$ be the set of centers of these stars (if $t_i = 1$, for the star K_{1,t_i} we pick the center arbitrarily). Let $L := V(H) \setminus C$ and note that every vertex in L is a leaf. We define an ordering of the leaves in L as follows. Given two leaves $\ell, m \in L$ we write $\ell < m$ if and only if $\ell u < mw$ in $H^{\leq}$, where $u, w \in C$ are the unique neighbors of ℓ and m in $H^{\leq}$ respectively (note u and w are not necessarily distinct). Set $L =: \{\ell_1, \ldots, \ell_{|L|}\}$ where $\ell_1 < \cdots < \ell_{|L|}$; note that $|L| = |E(H)|$.

We are now ready to embed $H^{\leq}$ into a middle increasing $\star$-canonical ordering of K_h. We first assume that the canonical part of K_h is a min or an inverse min ordering. Then, using the labeling given in Definitions 2.2 and 2.7, it is easy to check that for every $1 \leq i < j < k, m \leq h-1$, we have

$$\begin{aligned} v_i v_k &< v_j v_m, \\ v_i v_k &< v_j x, \qquad \text{and} \\ v_i x &< v_j v_k \,. \end{aligned} \tag{3.1}$$

We embed the vertices in $L = \{\ell_1, \ldots, \ell_{|L|}\}$ into the $\star$-canonical ordering of K_h as follows:

$$\ell_i \mapsto v_i \text{ for every } i \in \big[|L|\big] \,.$$

We embed the vertices in C arbitrarily among the rest of the vertices in K_h. We need to check that this embedding induces a copy of $H^{\leq}$ in our edge-ordered K_h. This is clearly the case though: if $e_1, e_2 \in E(H^{\leq})$ such that $e_1 < e_2$ then e_1 is mapped to some edge $v_i y$ in K_h and e_2 to some edge $v_j z$ in K_h, where $i < j \leq |L|$. Then (3.1) implies that $v_i y < v_j z$ in our edge-ordering of K_h.

If the canonical part of K_h is a max or an inverse max ordering, then we proceed analogously. In this case we embed the leaves in L at the end of the $\star$-canonical ordering and the vertices in C at the beginning. More precisely, we define the embedding so that

$$\ell_i \mapsto v_{|C|+i-1} \text{ for every } i \in \big[|L|\big] \,,$$

and we embed the vertices in C arbitrarily among the rest of the vertices K_h. Then similarly to before, this embedding induces a copy of $H^{\leq}$ in our edge-ordering of K_h. □

There are some cases where the solution of Question 1.2 is an easy consequence of known tiling results for (unordered) graphs. In particular, the next result solves this problem for all edge-orderings of connected universally tileable graphs.

Proposition 3.1.

- *Let $K_3^{\leq}$ denote the edge-ordered version of K_3. Then $f(n, K_3^{\leq}) = 2n/3$.*
- *Let S denote an edge-ordered graph whose underlying graph is a star. Then $f(n, S) = n/2 + O(1)$.*
- *Let $P := 132$. Then $f(n, P) = n/2 + O(1)$.*

- *Let $P' := 213$. Then $f(n, P') = n/2 + O(1)$.*
- *Recall $P_3^{\leqslant} = 123$. Then $f(n, P_3^{\leqslant}) = n/2 + o(n)$.*

Proof. The first part of the proposition follows immediately from the Corrádi–Hajnal theorem [8].

Up to isomorphism, there is only one edge-ordering of a star on a given number of vertices. Thus, for any edge-ordered star S, the Kühn–Osthus theorem [16] implies that $f(n, S) = n/2 + O(1)$.

Any edge-ordering of C_4 contains a copy of the edge-ordered path $P = 132$. The Kühn–Osthus theorem [16] implies that the minimum degree threshold for forcing a perfect C_4-tiling in an n-vertex graph G is $n/2+O(1)$; so $f(n, P) \leq n/2+O(1)$. Moreover, consider the n-vertex graph consisting of two disjoint cliques X, Y whose sizes are as equal as possible, under the constraint that 4 does not divide $|X|$ or $|Y|$. Then every edge-ordering G of this graph does not contain a perfect P-tiling and $\delta(G) \geq n/2 - 2$. Thus, $f(n, P) > n/2 - 2$ and so $f(n, P) = n/2 + O(1)$. The same argument shows that $f(n, P') = n/2 + O(1)$. Finally, in Theorem 1.1 we saw that $f(n, P_3^{\leqslant}) = (1/2 + o(1))n$. □

4. Proof of Theorem 1.6

Let G be an edge-ordered graph on $n \geq T(F)$ vertices with minimum degree $\delta(G) \geq (1 - \frac{1}{T(F)})n$, and so that $|F|$ divides n. Let G' denote the underlying graph of G. When $T(F)$ divides n, we apply the Hajnal–Szemerédi theorem [13] to G', to obtain an (unordered) perfect $K_{T(F)}$-tiling in G'. By the definition of $T(F)$, each edge-ordered copy of $K_{T(F)}$ in G contains a perfect F-tiling. Thus, combining these tilings, we obtain a perfect F-tiling in G.

When $T(F)$ does not divide n, $n = aT(F) + b$ for $a, b \in \mathbb{N}$ such that $0 < b < T(F)$. As n and $T(F)$ are divisible by $|F|$, we have that $b/|F| \in \mathbb{N}$. Since $b/T(F) < 1$, we must have that $\delta(G) \geq n - a = (1 - \frac{1}{T(F)})(n - b) + b$.

We will now repeatedly remove disjoint copies of F from G, until the resulting edge-ordered graph has its order divisible by $T(F)$. Assume that we have already removed c copies of F from G, where $0 \leq c < b/|F|$; then the remaining edge-ordered graph on $n - c|F|$ vertices has minimum degree at least

$$\Big(1 - \frac{1}{T(F)}\Big)(n - b) + b - c|F| \geq \Big(1 - \frac{1}{T(F)}\Big)(n - c|F|) + (b - c|F|)\frac{1}{T(F)}.$$

This lower bound guarantees that an unordered $K_{T(F)}$ exists in the underlying graph; within the corresponding edge-ordered copy of $K_{T(F)}$ lying in G, we can find a copy of F. Thus, we may again remove a copy of F and repeat this process.

This process ensures that we can remove $b/|F|$ copies of F from G. The resulting edge-ordered graph has $n - b$ vertices and minimum degree at least $(1 - \frac{1}{T(F)})(n - b)$. Since $T(F)$ divides $n - b$, as in the previous case this edge-ordered graph contains a perfect F-tiling; combining this tiling with our removed copies of F, we obtain a perfect F-tiling in G, as desired. □

5. Proof of Theorem 1.1

For the proof of Theorem 1.1 we use the absorbing method, which divides the proof into two main parts: finding an absorber and constructing an almost perfect $P_k^{\leqslant}$-tiling.

The following two subsections are devoted to the Absorbing Lemma (Lemma 5.5) and the Almost Perfect Tiling Lemma (Lemma 5.7) respectively. We finish this section by combining these two results to give the proof of Theorem 1.1.

5.1. Absorbers.

Let F be an edge-ordered graph. Given an edge-ordered graph G, a set $S \subseteq V(G)$ is an *F-absorbing set for $Q \subseteq V(G)$*, if both $G[S]$ and $G[S \cup Q]$ contain perfect F-tilings.

To prove Theorem 1.1, we make use of the following, now standard, absorbing lemma.

Lemma 5.1. *Let $f, s \in \mathbb{N}$ and $\xi > 0$. Suppose that F is an edge-ordered graph on f vertices. Then there exists an $n_0 \in \mathbb{N}$ such that the following holds. Suppose that G is an edge-ordered graph on $n \geq n_0$ vertices so that, for any $x, y \in V(G)$, there are at least ξn^{sf-1} $(sf-1)$-sets $X \subseteq V(G)$ such that both $G[X \cup \{x\}]$ and $G[X \cup \{y\}]$ contain perfect F-tilings. Then $V(G)$ contains a set M so that*

- *$|M| \leq (\xi/2)^f n/4$;*
- *M is an F-absorbing set for any $W \subseteq V(G) \setminus M$ such that $|W| \leq (\xi/2)^{2f} n/(32s^2f^3)$ and $|W| \in f\mathbb{N}$.* □

Lemma 5.1 was proven by Lo and Markström [17, Lemma 1.1] in the case when G is an unordered graph. However, the proof in the edge-ordered setting is identical (so we do not provide a proof here).

As mentioned in the introduction, Rödl [20] proved that every edge-ordered graph on n vertices and at least $k(k+1)n/2$ edges contains a monotone path of length k. Here we will need the following supersaturated version of this result.

Lemma 5.2 (Supersaturation Lemma)**.** *Let $k \in \mathbb{N}$ and $\zeta > 0$. Then there exists an $n_0 \in \mathbb{N}$ such that the following holds for every $n \geq n_0$. Every n-vertex edge-ordered graph G with at least ζn^2 edges contains at least $\zeta^k 2^{-k^2} n^{k+1}$ copies of $P_k^{\leqslant}$.*

Proof. The proof goes by induction on k. The case $k = 1$ is trivial. Suppose the statement is true for $k-1$, and take n_0 large enough to apply the induction hypothesis for $\zeta/2$.

Let G be an n-vertex edge-ordered graph as in the statement of the lemma. For every vertex $v \in V(G)$ delete the last $\min\{d(v), \zeta n/2\}$ edges incident to v. Let $\tilde{G}$ denote the resulting edge-ordered graph. Since $e(\tilde{G}) \geq \zeta n^2 - \zeta n^2/2 = \zeta n^2/2$, by the induction hypothesis we have that $\tilde{G}$ contains at least

$$\left(\frac{\zeta}{2}\right)^{k-1} \cdot 2^{-(k-1)^2} n^k = \zeta^{k-1} 2^{-(k-1)^2-(k-1)} n^k$$

copies of $P_{k-1}^{\leqslant}$. Fix one such copy $P = v_1 \cdots v_k$ and observe that, since $d_{\tilde{G}}(v_k) > 0$, $\zeta n/2$ edges incident to v_k were deleted from G that are all larger than $v_{k-1}v_k$ in the total order of $E(G)$. Moreover, at most $k-1$ of them are incident to a vertex in P, which implies that at least $\zeta n/2 - (k-1) \geq \zeta n/4$ of them, combined with P, form a copy of $P_k^{\leqslant}$ in G. Therefore, we obtain at least

$$\frac{\zeta^{k-1}}{2^{(k-1)^2+(k-1)}} n^k \cdot \frac{\zeta}{4} n \geq \frac{\zeta^k}{2^{k^2}} n^{k+1}$$

copies of $P_k^{\leqslant}$ in G. □

Note the proof of Lemma 5.2 really uses that the path we consider is monotone. Indeed, the inductive step our proof requires that given an edge-ordered path P, we add an edge e larger than all those edges in P, and that e is incident to the largest edge currently in P.

In order to apply Lemma 5.1 we introduce the following notion.

Definition 5.3 (Local Absorbers)**.** Let $x, y \in V(G)$ be distinct vertices of an edge-ordered graph G. Given disjoint $P_x, P_y \in \binom{V(G)}{k}$ and a vertex $w \in V(G) \setminus (P_x \cup P_y)$, we say that the set

$$A := P_x \cup P_y \cup \{w\}$$

is a *$P_k^{\leqslant}$-local-absorber for x and y* if

(1) $G[\{x\} \cup P_x]$ and $G[\{w\} \cup P_x]$ contain spanning copies of $P_k^{\leqslant}$ and
(2) $G[\{y\} \cup P_y]$ and $G[\{w\} \cup P_y]$ contain spanning copies of $P_k^{\leqslant}$.

Observe that if A is a $P_k^{\leqslant}$-local-absorber for x and y then both $G[A \cup \{x\}]$ and $G[A \cup \{y\}]$ contain perfect $P_k^{\leqslant}$-tilings. That is, A can play the role of X in Lemma 5.1 with $s = 2$. The following lemma allows us to find many local absorbers for every pair of vertices $x, y \in V(G)$.

Lemma 5.4. *For every $k \in \mathbb{N}$ and for every $0 < \eta < 1/2$ there is a $\xi > 0$ and an $n_0 \in \mathbb{N}$ such that the following holds for every $n \geq n_0$. Let G be an n-vertex edge-ordered graph with $\delta(G) \geq (1/2+\eta)n$. Then for every two vertices $x, y \in V(G)$ there are at least ξn^{2k+1} $P_k^{\leqslant}$-local-absorbers for x and y.*

Proof. Given $k \in \mathbb{N}$ and $\eta > 0$ let

$$\zeta := \frac{\eta^k}{2^{k^2+4k}} \qquad \text{and} \qquad \xi := \frac{\eta \zeta^2}{16(2k+1)!}\,,$$

and suppose $n_0 \in \mathbb{N}$ is sufficiently large. Let G be as in the statement of the lemma.

For every $x \in V(G)$ define

$$\mathcal{P}_x := \left\{ P \subseteq \binom{V(G)}{k} : G[\{x\} \cup P] \text{ contains a copy of } P_k^{\leqslant} \right\}.$$

We first show that there is a subset $\mathcal{P}'_x \subseteq \mathcal{P}_x$ of size at least $\zeta n^k/2$ such that for every $P \in \mathcal{P}'_x$ there is a set $W_x(P) \subseteq V(G) \setminus P$ satisfying

(i) $P \in \mathcal{P}_w$ for every $w \in W_x(P)$ and
(ii) $|W_x(P)| \geq \left(\frac{1}{2} + \frac{\eta}{4}\right)n$.

In order to do this, we partition $N(x) = L(x) \dot\cup S(x)$ as follows. We say a vertex $u \in N(x)$ is *large* if the set $\{v \in N(u) : xu < vu\}$ is of size at least $\eta n/2$. Otherwise, we say u is *small*. Let $L(x)$ and $S(x)$ denote the set of large and small vertices in $N(x)$, respectively. Notice that if u is small then the set $\{v \in N(u) : xu > vu\}$ is of size at least $\eta n/2$ (and actually, at least of size $n/2$). Assume that $|L(x)| \geq |N(x)|/2 \geq n/4$; the case $|S(x)| \geq n/4$ is analogous.

For every vertex $u \in L(x)$, let $E(u)$ be the set of the last $\eta n/2$ edges incident to u in the total order of $E(G)$. Since u is large, all edges in $E(u)$ are larger than xu. For $E_x := \bigcup_{u \in L(x)} E(u)$, consider the subgraph $\widetilde{G} := (V(G), E_x) \subseteq G$. Note that $|E_x| \geq \eta n^2/16$. Thus, Lemma 5.2 implies that $\widetilde{G}$ contains at least ζn^{k+1} monotone paths of length k. Since every edge in E_x is incident to a vertex in $L(x)$, by dropping the first or the last vertex in each path, we obtain at least $\zeta n^k/2$ monotone paths of length $k-1$ in $\widetilde{G}$ starting with a vertex in $L(x)$. That is, the set

$$\mathcal{P}'_x := \left\{ P \subseteq \binom{V(G)}{k} : \widetilde{G}[P] \text{ contains a copy of } P_{k-1}^{\leqslant} \text{ starting with a vertex in } L(x) \right\}$$

is of size at least $\zeta n^k/2$. Moreover, notice that $\mathcal{P}'_x \subseteq \mathcal{P}_x$. Indeed, let $u_1 \cdots u_k$ be a monotone path with $P = \{u_1, \ldots, u_k\} \in \mathcal{P}'_x$. Since $u_1 \in L(x)$, we have $xu_1 < u_1u_2$, and therefore $G[\{x\} \cup P]$ contains a copy of $P_k^{\leqslant}$, meaning that $P \in \mathcal{P}_x$. Now, we shall prove that for every $P \in \mathcal{P}'_x$ there is a set $W_x(P)$ satisfying (i) and (ii).

Consider some $P = \{u_1, \ldots, u_k\} \in \mathcal{P}'_x$ where u_1u_2 is the first edge of the copy of $P_{k-1}^{\leqslant}$ in $\widetilde{G}[P]$. Let $N'(u_1)$ denote the set of vertices w in $N(u_1)$ such that $u_1w \notin E(u_1)$. Define $W_x(P) := N'(u_1) \setminus P$. Thus, since $u_1u_2 \in E(u_1)$, for $w \in W_x(P)$ we have $wu_1 < u_1u_2$ which means that $W_x(P)$ satisfies condition (i). Condition (ii) follows as $\delta(G) \geq (1/2+\eta)n$ and $|E(u_1)| = \eta n/2$.

Finally, given $x, y \in V(G)$ consider $\mathcal{P}'_x$ and $\mathcal{P}'_y$. Observe that the number of pairs $(P_x, P_y) \in \mathcal{P}'_x \times \mathcal{P}'_y$ such that $|P_x \cap P_y| \geq 1$ is at most $k^2 n^{2k-1}$ and therefore, since n is sufficiently large, there are at least

$$\frac{|\mathcal{P}'_x \times \mathcal{P}'_y|}{2} \geq \frac{\zeta^2 n^{2k}}{8}$$

disjoint pairs in $\mathcal{P}'_x \times \mathcal{P}'_y$. Given a disjoint pair $(P_x, P_y) \in \mathcal{P}'_x \times \mathcal{P}'_y$ and a vertex $w \in W_x(P_x) \cap W_y(P_y)$, it is easy to see that $A := P_x \cup P_y \cup \{w\}$ is a $P_k^{\leqslant}$-local-absorber for x and y. Because of (ii),

$|W_x(P_x) \cap W_y(P_y)| \geq \eta n/2$, and therefore, there are at least

$$\frac{\zeta^2 n^{2k}}{8} \cdot \frac{\eta n}{2} \cdot \frac{1}{(2k+1)!} = \xi n^{2k+1}$$

$P_k^{\leqslant}$-local-absorbers for x and y. In particular, we divide by $(2k+1)!$ as the same $P_k^{\leqslant}$-local-absorber A arises from at most $(2k+1)!$ tuples (P_x, P_y, w). □

The Absorbing Lemma is now an immediate consequence of Lemmas 5.1 and 5.4.

Lemma 5.5 (Absorbing Lemma)**.** *For every $k \in \mathbb{N}$ and $\eta > 0$ there is $0 < \xi < \eta$ and an $n_0 \in \mathbb{N}$ such that the following holds for every $n \geq n_0$. If G is an edge-ordered graph on n vertices with $\delta(G) \geq (1/2 + \eta)n$, then there is a set $M \subseteq V(G)$ of size at most ξn which is a $P_k^{\leqslant}$-absorbing set for every $W \subseteq V(G) \setminus M$ such that $|W| \in (k+1)\mathbb{N}$ and $|W| \leq \xi^3 n$.* □

5.2. **Almost perfect tilings.** Given an (unordered) graph F, Komlós [14] established an asymptotically optimal minimum degree condition that forces a graph G to contain an F-tiling covering all but at most $o(n)$ vertices. To present this result, we need to introduce the following parameter. Given a graph F, the *critical chromatic number $\chi_{cr}(F)$ of F* is defined as

$$\chi_{cr}(F) := (\chi(F) - 1)\frac{|V(F)|}{|V(F)| - \sigma(F)},$$

where $\chi(F)$ is the chromatic number of F and $\sigma(F)$ denotes the size of the smallest possible color class in any $\chi(F)$-coloring of F.

Theorem 5.6 ([14])**.** *For every $\varepsilon > 0$ and every graph F, there is an $n_0 \in \mathbb{N}$ such that the following holds for every $n \geq n_0$. If G is a graph on n vertices with*

$$\delta(G) \geq \Big(1 - \frac{1}{\chi_{cr}(F)}\Big)n,$$

then G contains an F-tiling covering at least $(1 - \varepsilon)n$ vertices.

Theorem 5.6 is best possible in the following sense: given any graph F and any $\gamma < 1 - \frac{1}{\chi_{cr}(F)}$, there exist $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ so that if $n \geq n_0$ there is an n-vertex graph G with $\delta(G) \geq \gamma n$ that does not contain an F-tiling covering at least $(1 - \varepsilon)n$ vertices.

For the (unordered) path P_k of length k, Theorem 5.6 ensures the existence of an almost perfect P_k-tiling in every n-vertex graph with minimum degree $\delta(G) \geq n/2$ when k is odd and $\delta(G) \geq kn/(2k+2)$ when k is even. The following lemma says that the same minimum degree condition ensures an almost perfect $P_k^{\leqslant}$-tiling in an edge-ordered graph G.

Lemma 5.7 (Almost Perfect Tiling Lemma)**.** *Let $k \in \mathbb{N}$ and $\varepsilon > 0$. There is an $n_0 \in \mathbb{N}$ such that the following holds for every $n \geq n_0$. Let G be an n-vertex edge-ordered graph with*

$$\delta(G) \geq \begin{cases} \frac{n}{2} & \text{if } k \text{ is odd} \\ \frac{kn}{2k+2} & \text{if } k \text{ is even}. \end{cases}$$

Then, G contains a $P_k^{\leqslant}$-tiling covering at least $(1 - \varepsilon)n$ vertices.

The same example that shows Theorem 5.6 is best possible for P_k shows that Lemma 5.7 is best possible for $P_k^{\leqslant}$. More precisely, if k is odd consider any $0 < \gamma < 1/2$ and set $\varepsilon := 1/2 - \gamma$; if k is even consider any $0 < \gamma < k/(2k+2)$ and set $\varepsilon := k/(2k+2) - \gamma$. Let G be any edge-ordering of the complete bipartite graph with vertex classes of size γn and $(1 - \gamma)n$. Then $\delta(G) = \gamma n$ and G does not contain a $P_k^{\leqslant}$-tiling covering more than $(1 - \varepsilon)n$ vertices.

Proof of Lemma 5.7. Given $k \in \mathbb{N}$ and $\varepsilon > 0$, let $\zeta := \frac{(k+1)^2-1}{4(k+1)^2}$ and let $n_1 \in \mathbb{N}$ be the n_0 given by Lemma 5.2 for $k+1$ instead of k. Moreover, let $m \geq \frac{2n_1}{\varepsilon(k+1)}$ and suppose n_0 is sufficiently large with respect to all other constants. Finally, let G be as in the statement of the lemma.

Set $a := \lceil (k+1)/2 \rceil$ and $b := \lfloor (k+1)/2 \rfloor$, and notice that $\chi_{cr}(P_k) = \chi_{cr}(K_{am,bm})$. Therefore, applying Theorem 5.6 (to the underlying graph of G) we obtain a $K_{am,bm}$-tiling covering at least $(1-\varepsilon/2)n$ vertices. We shall prove that in each $K_{am,bm}$ there is a $P_k^{\leqslant}$-tiling covering all but at most n_1 vertices. Observe that, for every positive integer $t \in \mathbb{N}$, we have

$$|E(K_{at,bt})| \geq \frac{(k+1)^2-1}{4} t^2 = \zeta (k+1)^2 t^2 = \zeta |V(K_{at,bt})|^2 \,. \tag{5.1}$$

Moreover, $|V(K_{am,bm})| = (a+b)m = (k+1)m \geq n_1$, and therefore we may apply Lemma 5.2. In fact, we will apply Lemma 5.2 iteratively to find the desired $P_k^{\leqslant}$-tiling in $K_{am,bm}$.

If k is even, then we apply Lemma 5.2 to find a copy of $P_{k+1}^{\leqslant}$ in $K_{am,bm}$. After deleting one vertex, we get a copy of $P_k^{\leqslant}$ with exactly $a = (k+2)/2$ vertices in the class of size am. If k is odd, then we apply Lemma 5.2 to obtain a copy of $P_k^{\leqslant}$, which must contain exactly $a = (k+1)/2$ vertices in the class of size am. In both cases, removing this copy of P_k from $K_{am,bm}$ results in a copy of $K_{a(m-1),b(m-1)}$. Thus, since (5.1) holds for every $t \in \mathbb{N}$, we may iteratively apply Lemma 5.2 to find vertex-disjoint copies of $P_k^{\leqslant}$ in $K_{am,bm}$ until there are at most n_1 vertices left (in each $K_{am,bm}$).

The initial $K_{am,bm}$-tiling has at most $n/|V(K_{am,bm})| = n/(m(k+1))$ copies of $K_{am,bm}$ covering at least $(1-\varepsilon/2)n$ vertices in G. Each of these copies of $K_{am,bm}$ has a $P_k^{\leqslant}$-tiling covering all but at most n_1 vertices. Therefore, there is a $P_k^{\leqslant}$-tiling in G covering all but at most

$$\frac{\varepsilon n}{2} + \frac{n}{m(k+1)} n_1 \leq \varepsilon\, n$$

vertices, where the last inequality follows as $\frac{n_1}{m(k+1)} \leq \frac{\varepsilon}{2}$. □

5.3. **Proof of Theorem 1.1.** To prove the 'moreover' part, given any $n \in \mathbb{N}$ divisible by $k+1$, let G_0 be an n-vertex edge-ordered graph consisting of two disjoint cliques whose sizes are as equal as possible under the constraint that neither has size divisible by $k+1$. Thus, G_0 does not contain a perfect $P_k^{\leqslant}$-tiling and $\delta(G_0) \geq \lfloor n/2 \rfloor - 2$.

Given $k \in \mathbb{N}$ and $\eta > 0$, let $0 < \xi < \eta$ be given by Lemma 5.5. Let $n_0 \in \mathbb{N}$ be sufficiently large and let G be as in the statement of the theorem. Lemma 5.5 yields a set $M \subseteq V(G)$ of size at most $\xi n \leq \eta n$ which is a $P_k^{\leqslant}$-absorbing set for every $W \subseteq V(G) \setminus M$ such that $W \in (k+1)\mathbb{N}$ and $|W| \leq \xi^3 n$. As $\delta(G \setminus M) \geq n/2 + \eta n - \xi n \geq n/2$, Lemma 5.7 implies $G \setminus M$ contains a $P_k^{\leqslant}$-tiling $\mathcal{T}_1$ covering all but at most $\xi^3 n$ vertices. Let L denote the set of vertices not covered by this tiling; notice that as $|G|$ and $|M|$ are divisible by $k+1$, so is $|L|$. By definition of M, $G[M \cup L]$ contains a perfect $P_k^{\leqslant}$-tiling $\mathcal{T}_2$. Thus, $\mathcal{T}_1 \cup \mathcal{T}_2$ is a perfect $P_k^{\leqslant}$-tiling in G. □

Remark 5.8. Recall that, for $k \geq 4$, there is always an edge-ordering of P_k that is not tileable. It would, however, be interesting to determine which edge-orderings of P_k one can extend Theorem 1.1 to cover. Notice that our proof of Theorem 1.1 is tailored to monotone paths though.

Indeed, the proof of Lemma 5.7 uses Lemma 5.2, whose proof is specific to monotone paths $P_k^{\leqslant}$. Further, in the proof of Lemma 5.4, we use the fact that if $P = u_1 \cdots u_{k+1}$ is a monotone path, then $u_1 \cdots u_k$ is isomorphic to $u_2 \cdots u_{k+1}$. In other words, the path obtained by dropping the last vertex is isomorphic to the one obtained by dropping the first one. It is not hard to see that this property is satisfied only by monotone paths.

In a forthcoming paper, the second and third authors will explore a more general strategy for establishing minimum degree thresholds for perfect tilings in edge-ordered graphs.

6. Concluding remarks

In this paper we have characterized those edge-ordered graphs that are tileable; similarly to the characterization of Turánable edge-ordered graphs, the tileable edge-ordered graphs F are those that can be embedded in specific orderings – which we call the $\star$-canonical orderings – of the complete graph $K_{|F|}$. For the characterization of Turánable graphs, namely Theorem 2.4, all four canonical orderings are necessary in the following sense: for every $n \geq 4$ and every canonical ordering $K_n^{\leq}$ of K_n, there is a non-Turánable edge-ordered n-vertex graph F such that F can be embedded into all the canonical orderings of K_n other than $K_n^{\leq}$. Thus, it is natural to raise the following question.

Question 6.1. *Are all twenty $\star$-canonical orderings necessary in Theorem 2.6? That is, does Theorem 2.6 still hold if we omit some of the $\star$-canonical orderings from the statement?*

From a computer-assisted check, we know that at least the following *eight* $\star$-canonical orderings are necessary: smaller increasing/decreasing of types min/inverse min, and larger increasing/decreasing of types max/inverse max. Note that these include the four canonical orderings.

In this paper we have also answered Question 1.2 in the case of monotone paths and for a few other special types of edge-ordered graph. Recall that in Section 5.2 we computed the minimum degree threshold for an edge-ordered graph to contain an almost perfect $P_k^{\leq}$-tiling. It is also natural to consider this problem more generally. This motivates the following definition.

Definition 6.2 (Almost tileable)**.** *An edge-ordered graph F is* almost tileable *if for every $0 < \varepsilon < 1$ there exists a $t \in \mathbb{N}$ such every edge-ordering of the graph K_t contains an F-tiling covering all but at most εt vertices of K_t.*

It is easy to see that this notion is equivalent to being Turánable.

Proposition 6.3. *An edge-ordered graph F is almost tileable if and only if F is Turánable.*

Proof. The forwards direction is immediate. For the reverse direction, consider any F that is Turánable. Given any $0 < \varepsilon < 1$ define $t := \lceil T(F)/\varepsilon \rceil$. (Recall $T(F)$ is defined in the statement of Theorem 1.6.) Then given any edge-ordering of K_t, by definition of $T(F)$ we may repeatedly find vertex-disjoint copies of F in K_t until we have covered all but fewer than $T(F)$ vertices in K_t. That is, we have an F-tiling covering all but at most εt vertices of K_t, as desired. □

In light of Proposition 6.3 we propose the following question.

Question 6.4. *Let F be a fixed Turánable edge-ordered graph. What is the minimum degree threshold for forcing an almost perfect F-tiling in an edge-ordered graph on n vertices? More precisely, given any $\varepsilon > 0$, what is the minimum degree required in an n-vertex edge-ordered graph G to force an F-tiling in G covering all but at most εn vertices?*

We emphasize that just because the notions of Turánable and almost tileable are equivalent, this certainly does not mean that the answer to Question 6.4 will be the 'same' as the Turán threshold. For example, whilst Rödl [20] showed that one only requires $k(k+1)n/2$ edges in an n-vertex edge-ordered graph G to force a copy of $P_k^{\leq}$, Lemma 5.7 implies G must be much denser to contain an almost perfect $P_k^{\leq}$-tiling.

Recall every Turánable (and therefore tileable) edge-ordered graph F does not contain a copy of K_4. We are unaware, however, of any result that forbids F from having large chromatic number.

Question 6.5. *Is it true that for every $k \in \mathbb{N}$ there is a Turánable edge-ordered graph F whose underlying graph has chromatic number at least k?*

Recall that due to Proposition 2.14, given a Turánable edge-ordered graph F we can construct a tileable graph by adding two suitable new vertices of degree one. Thus, Question 6.5 is equivalent to the following question.

Question 6.6. *Is it true that for every $k \in \mathbb{N}$ there is a tileable edge-ordered graph G whose underlying graph has chromatic number at least k?*

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