TILING PROBLEMS IN EDGE-ORDERED GRAPHS

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Abstract. Given graphs F and G, a perfect F-tiling in G is a collection of vertex-disjoint copies of F in G that together cover all the vertices in G. The study of the minimum degree threshold forcing a perfect F-tiling in a graph G has a long history, culminating in the Kühn–Osthus theorem [Combinatorica 2009] which resolves this problem, up to an additive constant, for all graphs F. We initiate the study of the analogous question for edge-ordered graphs. In particular, we characterize for which edge-ordered graphs F this problem is well-defined. We also apply the absorbing method to asymptotically determine the minimum degree threshold for forcing a perfect P-tiling in an edge-ordered graph, where P is any fixed monotone path.

1. Introduction

1.1. Monotone paths in edge-ordered graphs.

An *edge-ordered graph* G is a graph equipped with a total order $\leq$ of its edge set $E(G)$. Usually we think of a total order of $E(G)$ as a labeling of the edges with labels from $\mathbb{R}$, where the labels inherit the total order of $\mathbb{R}$ and where edges are assigned distinct labels. A path P in G is *monotone* if the consecutive edges of P form a monotone sequence with respect to $\leq$. We write $P_k^{\leqslant}$ for the monotone path of length k (i.e., on k edges).

The study of monotone paths in edge-ordered graphs dates back to the 1970s. Chvátal and Komlós [7] raised the following question: what is the largest integer $f(K_n)$ such that every edge-ordering of K_n contains a copy of the monotone path $P_{f(K_n)}^{\leqslant}$ of length $f(K_n)$? Over the years there have been several papers on this topic [4, 5, 6, 11, 17, 19]. In a recent breakthrough, Bucić, Kwan, Pokrovskiy, Sudakov, Tran, and Wagner [4] proved that $f(K_n) \geq n^{1-o(1)}$. The best known upper bound on $f(K_n)$ is due to Calderbank, Chung, and Sturtevant [6] who proved that $f(K_n) \leq (1/2 + o(1))n$. There have also been numerous papers on the wider question of the largest integer $f(G)$ such that every edge-ordering of a graph G contains a copy of a monotone path of length $f(G)$. See the introduction of [4] for a detailed overview of the related literature.

A classical result of Rödl [19] yields a Turán-type result for monotone paths: every edge-ordered graph with n vertices and with at least $k(k+1)n/2$ edges contains a copy of $P_k^{\leqslant}$. More recently, Gerbner, Methuku, Nagy, Pálvölgyi, Tardos, and Vizer [10] initiated the systematic study of the Turán problem for edge-ordered graphs.

It is also natural to seek conditions that force an edge-ordered graph G to contain a collection of vertex-disjoint monotone paths $P_k^{\leqslant}$ that cover all the vertices in G, that is, a *perfect $P_k^{\leqslant}$-tiling* in G. Our first result asymptotically determines the minimum degree threshold that forces a perfect $P_k^{\leqslant}$-tiling.

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Theorem 1.1. *Given any $k \in \mathbb{N}$ and $\eta > 0$, there exists an $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ where $(k+1)|n$ then the following holds: if G is an n-vertex edge-ordered graph with minimum degree*

$$\delta(G) \geq (1/2 + \eta)n$$

then G contains a perfect $P_k^{\leqslant}$-tiling. Moreover, for all $n \in \mathbb{N}$ with $(k+1)|n$, there is an n-vertex edge-ordered graph G_0 with $\delta(G_0) \geq \lfloor n/2 \rfloor - 2$ that does not contain a perfect $P_k^{\leqslant}$-tiling.

For the edge-ordered graph G_0 in Theorem 1.1, one can take any edge-ordering of the n-vertex graph consisting of two disjoint cliques whose sizes are as equal as possible under the constraint that neither has size divisible by $k+1$. Our proof of Theorem 1.1 in [3] provides the first application of the so-called *absorbing method* in the setting of edge-ordered graphs.

1.2. **The general problem.** Let F and G be edge-ordered graphs. We say that G *contains* F if F is isomorphic to a subgraph F' of G; here, crucially, the total order of $E(F)$ must be the same as the total order of $E(F')$ that is inherited from the total order of $E(G)$. In this case we say F' is a *copy of F in G*. For example, if G contains a path F' of length 3 with consecutive edges labeled 5, 17 and 4 then F' is a copy of the path F of length 3 with consecutive edges labeled 2, 3 and 1.

Given edge-ordered graphs F and G, an *F-tiling* in G is a collection of vertex-disjoint copies of F in G; an F-tiling in G is *perfect* if it covers all the vertices in G. In light of Theorem 1.1 we raise the following general question.

Question 1.2. *Let F be a fixed edge-ordered graph on $f \in \mathbb{N}$ vertices and let $n \in \mathbb{N}$ be divisible by f. What is the smallest integer $f(n, F)$ such that every edge-ordered graph on n vertices and of minimum degree at least $f(n, F)$ contains a perfect F-tiling?*

Theorem 1.1 implies that $f(n, P_k^{\leqslant}) = (1/2+o(1))n$ for all $k \in \mathbb{N}$. Note that the *unordered* version of Question 1.2 had been well-studied since the 1960s (see, e.g., [1, 8, 12, 14, 15]) and forty-five years later a complete solution, up to an additive constant term, was obtained via a theorem of Kühn and Osthus [15]. Very recently, the *vertex-ordered graph* version of this problem has been asymptotically resolved [2, 9].

Question 1.2 has a rather different flavor to its graph and vertex-ordered graph counterparts. In particular, there are edge-ordered graphs F for which, given *any* $n \in \mathbb{N}$, there exists an edge-ordering $\leq$ of the complete graph K_n that does not contain a single copy of F. Thus, for such F, Question 1.2 is trivial in the sense that clearly there is no minimum degree threshold $f(n, F)$ for forcing a perfect F-tiling. This motivates Definitions 1.3 and 1.4 below.

Definition 1.3 (Turánable)**.** An edge-ordered graph F is *Turánable* if there exists a $t \in \mathbb{N}$ such every edge-ordering of the graph K_t contains a copy of F.

Definition 1.4 (Tileable)**.** An edge-ordered graph F on f vertices is *tileable* if there exists a $t \in \mathbb{N}$ divisible by f such that every edge-ordering of the graph K_t contains a perfect F-tiling.

The following Ramsey-type result, attributed to Leeb (see [10, 18]), says that in every sufficiently large edge-ordered complete graph we must always find a subgraph which is *canonically ordered*. For $n \geq 5$ there are four non-isomorphic canonical edge-orderings of K_n. We omit the definitions of the canonical edge-orderings in this extended abstract, but they can be found in [10, Section 2.1].

Proposition 1.5. *For every $k \in \mathbb{N}$ there is an $m \in \mathbb{N}$ such that every edge-ordered complete graph K_m contains a copy of K_k that is canonically edge-ordered.*

In [10] it was observed that Proposition 1.5 yields the following full characterization of Turánable graphs.

Theorem 1.6 (Turánable characterization)**.** *An edge-ordered graph F on n vertices is Turánable if and only if all four canonical edge-orderings of K_n contain a copy of F.*

In [3], we prove a result analogous to Theorem 1.6 for tileable graphs. More precisely, we provide a full characterization of all n-vertex tileable graphs with respect to twenty fixed edge-orderings of the complete graph K_n. We call those orderings $\star$-*canonical orderings* of K_n. The full definition of the $\star$-canonical orderings is a little involved and we omit the details here, but the precise description can be found in [3].

Theorem 1.7 (Tileable characterization). *An edge-ordered graph F on n vertices is tileable if and only if all twenty $\star$-canonical orderings of K_n contain a copy of F.*

In [3] we study several consequences of Theorems 1.6 and 1.7. In particular, we prove that the notions of Turánable and tileable are genuinely different. More precisely, we show that there are (infinitely many) edge-ordered graphs that are Turánable but not tileable.

In [10] it is proven that no edge-ordering of K_4 is Turánable and consequently, any edge-ordered graph containing a copy of K_4 is not Turánable and therefore not tileable. Thus, for an edge-ordered graph to be tileable it cannot be too 'dense'. We show in [3] that no edge-ordering of K_4^- is tileable.[1]

A graph H is *universally tileable* if for any given ordering of $E(H)$, the resulting edge-ordered graph is tileable. Similarly, we say that H is *universally Turánable* if given any edge-ordering of H, the resulting edge-ordered graph is Turánable. Using [10, Theorem 2.18], in [3] we characterize all those graphs H that are universally tileable.

Theorem 1.8. *Let H be a graph. The following are equivalent:*

(a) H is universally tileable;
(b) H is universally Turánable;
(c) (i) H is a star forest (possibly with isolated vertices),[2] or
(ii) H is a path on three edges together with a (possibly empty) collection of isolated vertices, or
(iii) H is a copy of K_3 together with a (possibly empty) collection of isolated vertices.

Moreover, in [3] we determine the asymptotic value of $f(n, F)$ in Question 1.2 for all connected universally tileable edge-ordered graphs F.

The characterization of tileable edge-ordered graphs given in Theorem 1.7 lays the ground for the systematic study of Question 1.2. The second and third authors will investigate this problem further in a forthcoming paper. Already though we can say something about this question. Indeed, an almost immediate consequence of the Hajnal–Szemerédi theorem [12] is the following result.

Theorem 1.9. *Let F be a tileable edge-ordered graph and let $T(F)$ be the smallest possible choice of $t \in \mathbb{N}$ in Definition 1.4 for F. Given any integer $n \geq T(F)$ divisible by $|F|$,*

$$f(n, F) \leq \left(1 - \frac{1}{T(F)}\right) n.$$

The proofs of Theorems 1.1, 1.7, 1.8, and 1.9 can be found in [3]. In the next section we outline the main ideas in the proof of Theorem 1.1.

2. Outline of the proof of Theorem 1.1

As mentioned above, the proof of Theorem 1.1 applies the *absorbing method*. This approach reduces the problem of finding a perfect $P_k^{\leqslant}$-tiling into two sub-tasks: (i) obtain an 'absorbing structure' A in the host graph G, and (ii) find a $P_k^{\leqslant}$-tiling covering almost all of the vertices in $G \setminus A$.

[1] Recall that K_t^- denotes the graph obtained from K_t by removing an edge.

[2] A *star forest* is a graph whose components are all stars.

This latter task is achieved via a relatively straightforward application of a result of Komlós [13]. The main task is therefore constructing the absorbing structure.

Roughly speaking, for an edge-ordered graph G, we say that a set of vertices $A \subseteq V(G)$ is a $P_k^{\leqslant}$*-absorber* if, for every sufficiently small set of vertices $W \subseteq V(G) \setminus A$ whose size is divisible by $k+1$, we have that $G[W \cup A]$ contains a perfect $P_k^{\leqslant}$-tiling.

We apply (an edge-ordered version of) a lemma by Lo and Markström [16, Lemma 1.1] which implies that in order to construct such a $P_k^{\leqslant}$-absorber we only need to find many so-called 'local absorbers' for every pair of vertices $x, y \in V(G)$. More precisely, a *local absorber for x and y* is a small set $L \subseteq V(G)$ with the property that both $G[L \cup \{x\}]$ and $G[L \cup \{y\}]$ contain perfect $P_k^{\leqslant}$-tilings.

To build up such local absorbers L for x and y, we prove a supersaturated version of the aforementioned result of Rödl: every edge-ordered graph with linear average degree contains 'many' copies of $P_k^{\leqslant}$. In particular, as our edge-ordered graph G has $\delta(G) \geq (1/2 + o(1))n$ this allows us to find many copies of $P_{k-1}^{\leqslant}$ in the neighborhood $N_G(v)$ of any vertex $v \in V(G)$. In fact, with some care, one can show the following stronger property: for every two different vertices $x, y \in V(G)$ there are many vertices $w \in V(G)$ so that (a) G contains many copies P_{xw} of $P_{k-1}^{\leqslant}$ for which x (resp. w) can be added to the start or end of P_{xw} to form a copy of $P_k^{\leqslant}$ in G, and (b) G contains many copies P_{yw} of $P_{k-1}^{\leqslant}$ for which y (resp. w) can be added to the start or end of P_{yw} to form a copy of $P_k^{\leqslant}$ in G.

This now gives us the structure we need to construct the local absorbers L for x and y. Indeed, for every choice of w, P_{xw} and P_{yw} above, we define a local absorber $L := V(P_{xw}) \cup V(P_{yw}) \cup \{w\}$. Properties (a) and (b) ensure each such L is indeed a local absorber for x and y, as desired.

Note that from the outline above it may not seem clear why our proof is specific to monotone paths, rather than other edge-orderings of paths. However, the details of the proof very much rely on our paths being monotone. For example, one crucial property we exploit is that if $P = u_1 \cdots u_{k+1}$ is a monotone path, then $u_1 \cdots u_k$ is isomorphic to $u_2 \cdots u_{k+1}$. In other words, the path obtained by dropping the last vertex is isomorphic to the one obtained by dropping the first one. It is not hard to see that this property is satisfied only by monotone paths.

3. Almost perfect tilings and open problems

As part of the proof of Theorem 1.1 in [3], we establish the minimum degree threshold that forces an 'almost perfect' $P_k^{\leqslant}$-tiling in an edge-ordered graph. It is also natural to consider this problem more generally. This motivates the following definition.

Definition 3.1 (Almost tileable)**.** *An edge-ordered graph F is* almost tileable *if for every $0 < \varepsilon < 1$ there exists a $t \in \mathbb{N}$ such every edge-ordering of the graph K_t contains an F-tiling covering all but at most εt vertices of K_t.*

It is easy to see that this notion is equivalent to being Turánable.

Proposition 3.2. *An edge-ordered graph F is almost tileable if and only if F is Turánable.*

Proof. The forward direction is immediate. For the reverse direction, consider any F that is Turánable. Given any $0 < \varepsilon < 1$ define $t := \lceil T(F)/\varepsilon \rceil$. (Recall $T(F)$ is defined in the statement of Theorem 1.9.) Then given any edge-ordering of K_t, by definition of $T(F)$ we may repeatedly find vertex-disjoint copies of F in K_t until we have covered all but fewer than $T(F)$ vertices in K_t. That is, we have an F-tiling covering all but at most εt vertices of K_t, as desired. □

In light of Proposition 3.2 we propose the following question.

Question 3.3. *Let F be a fixed Turánable edge-ordered graph. What is the minimum degree threshold for forcing an almost perfect F-tiling in an edge-ordered graph on n vertices? More precisely, given any $\varepsilon > 0$, what is the minimum degree required in an n-vertex edge-ordered graph G to force an F-tiling in G covering all but at most εn vertices?*

Finally, we know that every Turánable (and therefore tileable) edge-ordered graph F does not contain a copy of K_4. However, we are unaware of any result that forbids F from having large chromatic number.

Question 3.4. *Is it true that for every $k \in \mathbb{N}$ there is a Turánable edge-ordered graph F whose underlying graph has chromatic number at least k?*

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