TURÁN DENSITY OF CLIQUES OF ORDER FIVE IN 3-UNIFORM HYPERGRAPHS WITH QUASIRANDOM LINKS

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Abstract. We show that 3-uniform hypergraphs with the property that all vertices have a quasirandom link graph with density bigger than 1/3 contain a clique on five vertices. This result is asymptotically best possible.

§1. Introduction

We study extremal problems for 3-uniform hypergraphs and here, unless stated otherwise, a hypergraph will always be 3-uniform. Recall that given an integer n and a hypergraph F the extremal number $\mathrm{ex}(n, F)$ is the maximum number of hyperedges that an n-vertex hypergraph can have without containing a copy of F. It is well known that the sequence $\mathrm{ex}(n, F)/\binom{n}{3}$ converges and the limit defines the *Turán density* $\pi(F)$. Determining $\pi(F)$ is a central open problem in extremal combinatorics. In fact, even the case when F is a clique on four vertices is still unresolved and known as the 5/9-conjecture of Turán.

Erdős and Sós [4] suggested a variation restricting the problem only to those F-free hypergraphs that are *uniformly dense* among large sets of vertices. More precisely, given a hypergraph F, Erdős and Sós asked for the supremum $d \in [0, 1]$ such that there exist arbitrarily large F-free hypergraphs $H = (V, E)$ for which every linear sized subset of the vertices induces a hypergraph of density at least d. Extremal results for uniformly dense hypergraphs in that context were studied in [2, 5, 6, 9, 12, 13]. For hypergraphs there are several other notions of "uniform density" that are closely related to the theory of quasirandom hypergraphs (see, e.g., [1, 16]) and corresponding extremal results were studied in [10, 11, 14, 15]. Here, we shall focus on the following notion.

Definition 1.1. *For a hypergraph $H = (V, E)$ and reals $d \in [0, 1]$, $\eta > 0$, we say that H is $(\eta, d, \curlywedge)$-dense if for all P, $Q \subseteq V \times V$ we have*

$$e_{\curlywedge}(P, Q) = \Big|\Big\{\big((x, y), (y, z)\big) \in \mathcal{K}_{\curlywedge}(P, Q) \colon \{x, y, z\} \in E\Big\}\Big| \geqslant d\,\big|\mathcal{K}_{\curlywedge}(P, Q)\big| - \eta|V|^3, \qquad (1.1)$$

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where $\mathcal{K}_{\wedge}(P, Q) = \{((x, y), (y', z)) \in P \times Q : y = y'\}$.

For a fixed hypergraph F, we define the corresponding Turán density

$$\pi_{\wedge}(F) = \sup\{d \in [0,1] : \text{for every } \eta > 0 \text{ and } n \in \mathbb{N} \text{ there exists an } F\text{-free, } (\eta, d, \wedge)\text{-dense hypergraph with at least } n \text{ vertices}\}. \quad (1.2)$$

In [14] the last three authors obtained a general upper bound for $\pi_{\wedge}(K_\ell^{(3)})$, which turned out to be best possible for all $\ell \leqslant 16$ except for $\ell = 5$, 9, and 10.

Theorem 1.2. *For every integer* $t \geqslant 2$ *we have*

$$\pi_{\wedge}(K_{2^t}^{(3)}) \leqslant \frac{t-2}{t-1}.$$

Moreover, we have

$$\begin{aligned} 0 &= \pi_{\wedge}(K_4^{(3)}), \\ \tfrac{1}{3} \leqslant \pi_{\wedge}(K_5^{(3)}) \leqslant \tfrac{1}{2} &= \pi_{\wedge}(K_6^{(3)}) = \dots = \pi_{\wedge}(K_8^{(3)}), \\ \text{and} \quad \tfrac{1}{2} \leqslant \pi_{\wedge}(K_9^{(3)}) \leqslant \pi_{\wedge}(K_{10}^{(3)}) \leqslant \tfrac{2}{3} &= \pi_{\wedge}(K_{11}^{(3)}) = \dots = \pi_{\wedge}(K_{16}^{(3)}). \qquad \square \end{aligned}$$

Here we close the gap for $\pi_{\wedge}(K_5^{(3)})$ and show that the lower bound is best possible.

Theorem 1.3 (Main result)**.** *We have that*

$$\pi_{\wedge}(K_5^{(3)}) = \frac{1}{3}.$$

Theorem 1.3 has a consequence for hypergraphs with quasirandom links. For a hypergraph $H = (V, E)$ the *link graph* $L_H(x)$ of a vertex x is defined to be the graph with vertex set V and edge set $\{yz \in V^{(2)} : xyz \in E(H)\}$. Recall that for given $d \in [0,1]$ and $\delta > 0$ a graph $G = (V, E)$ is said to be (δ, d)-*quasirandom* if for every subset of vertices $X \subseteq V$ the number of edges $e(X)$ inside X satisfies

$$\left| e(X) - d\frac{|X|^2}{2} \right| \leqslant \delta |V|^2.$$

One can check that if all the vertices of a hypergraph H have a (δ, d)-quasirandom link graph, then H is $(f(\delta), d, \wedge)$-dense, where $f(\delta) \longrightarrow 0$ as $\delta \longrightarrow 0$. In fact, such hypergraphs even satisfy in addition a matching upper bound for $e_{\wedge}(P, Q)$ in (1.1) and, hence, having quasirandom links is a stronger property. However, the lower bound construction for $\pi_{\wedge}(K_5^{(3)})$ given below has quasirandom links with density 1/3 and, therefore, Theorem 1.3 yields an asymptotically optimal result for such hypergraphs.

Example 1.4. *For a map* $\psi : V^{(2)} \longrightarrow \mathbb{Z}/3\mathbb{Z}$ *we define the hypergraph* $H_\psi = (V, E)$ *by*

$$xyz \in E \qquad \Longleftrightarrow \qquad \psi(xy) + \psi(xz) + \psi(zy) \equiv 1 \pmod 3. \qquad (1.3)$$

Observe that for any set of five different vertices $U = \{u_1, u_2, u_3, u_4, u_5\}$ double counting yields the identity

$$\sum_{u_i u_j u_k \in U^{(3)}} \big(\psi(u_i u_j) + \psi(u_i u_k) + \psi(u_j u_k)\big) = 3 \sum_{u_i u_j \in U^{(2)}} \psi(u_i u_j)\,.$$

Since the second sum is zero modulo 3, at least one of the ten triples in the first sum fails to satisfy (1.3). Consequently, H_ψ is $K_5^{(3)}$-free for every map ψ.

Moreover, if ψ is chosen uniformly at random, then following the lines of the proof of [14, Proposition 13.1] *shows that for every fixed $\delta > 0$ and sufficiently large $|V|$ with high probability the hypergraph H_ψ has the property that all link graphs are $(\delta, 1/3)$-quasirandom.*

Summarising the discussion above we arrive at the following corollary, which in light of Example 1.4 is asymptotically best possible.

Corollary 1.5. *For every $\varepsilon > 0$ there exist $\delta > 0$ and an integer n_0 such that every hypergraph on at least n_0 vertices all of whose link graphs are $(\delta, 1/3 + \varepsilon)$-quasirandom contains a copy of $K_5^{(3)}$.* □

The proof of Theorem 1.3 is based on the regularity method for hypergraphs. More precisely, we shall address the corresponding problem for reduced hypergraphs $\mathcal{A}$ (see Proposition 2.4). The proof of Proposition 2.4 is based on a further reduction to the case, when there exists an underlying bicolouring of the pairs $V^{(2)}$, which corresponds to a bicolouring of the vertices in the reduced hypergraph $\mathcal{A}$ (see Proposition 2.6). Finally, we show that in the context of Theorem 1.3 such bicoloured reduced hypergraphs yield a $K_5^{(3)}$ (see Proposition 2.7). Sections 4 and 5 are devoted to the proofs of Propositions 2.6 and 2.7.

§2. Reduced hypergraphs and bicolourings

Similar as in [11–14] the proof of Theorem 1.3 utilises the regularity method for hypergraphs. This allows us to transfer the problem to an extremal problem for reduced hypergraphs, which play a similar rôle for hypergraphs as reduced graphs in applications of Szemerédi's regularity lemma for graphs.

Definition 2.1. *Given a set of indices I and pairwise disjoint, non-empty sets of vertices $\mathcal{P}^{ij}$ for every pair of indices $ij \in I^{(2)}$, let for every triple of distinct indices $ijk \in I^{(3)}$ a tripartite hypergraph $\mathcal{A}^{ijk}$ with vertex classes $\mathcal{P}^{ij}$, $\mathcal{P}^{ik}$, and $\mathcal{P}^{jk}$ be given.*

We call the $\binom{|I|}{2}$-partite hypergraph $\mathcal{A}$ defined by

$$V(\mathcal{A}) = \dot{\bigcup_{ij \in I^{(2)}}} \mathcal{P}^{ij} \qquad \text{and} \qquad E(\mathcal{A}) = \dot{\bigcup_{ijk \in I^{(3)}}} E(\mathcal{A}^{ijk})$$

a reduced *hypergraph with* index set I. *Moreover, we say $\mathcal{A}$ has* vertex classes $\mathcal{P}^{ij}$ *and* constituents $\mathcal{A}^{ijk}$.

In this work the index set I will often be an ordered set and we may assume $I \subseteq \mathbb{N}$. When we say that a reduced hypergraph is sufficiently large, we mean that its index set is sufficiently large. Theorem 1.3 concerns $\boldsymbol{\wedge}$-dense and K_5-free hypergraphs H and next we define the corresponding properties in the context of reduced hypergraphs.

Definition 2.2. *For $d \in [0,1]$ we say that a reduced hypergraph $\mathcal{A}$ with index set I is* $(d, \boldsymbol{\wedge})$-dense, *if for every $ijk \in I^{(3)}$ and all vertices $P^{ij} \in \mathcal{P}^{ij}$ and $P^{ik} \in \mathcal{P}^{ik}$ we have*

$$d\big(P^{ij}, P^{ik}\big) = \big|\{P^{jk} \in \mathcal{P}^{jk} \colon P^{ij}P^{ik}P^{jk} \in E(\mathcal{A}^{ijk})\}\big| \geqslant d\,|\mathcal{P}^{jk}|\,.$$

Definition 2.3. *We say a reduced hypergraph $\mathcal{A}$ with index set I* supports a clique $K_\ell^{(3)}$ *if there are an ℓ-element subset $J \subseteq I$ and vertices $P^{ij} \in \mathcal{P}^{ij}$ for every $ij \in J^{(2)}$ such that*

$$P^{ij}P^{ik}P^{jk} \in E(\mathcal{A}^{ijk})$$

for all $ijk \in J^{(3)}$.

With these concepts at hand, it follows from [10, Theorem 3.3] that the upper bound in Theorem 1.3 is a direct consequence of the following statement for reduced hypergraphs.

Proposition 2.4. *For every $\varepsilon > 0$ every sufficiently large $\big(\frac{1}{3} + \varepsilon, \boldsymbol{\wedge}\big)$-dense reduced hypergraph $\mathcal{A}$ supports a $K_5^{(3)}$.*

The proof of Proposition 2.4 proceeds by contradiction, so we assume that for some $\varepsilon > 0$ there are $(\frac{1}{3} + \varepsilon, \boldsymbol{\wedge})$-dense reduced hypergraphs of unbounded size that do not support $K_5^{(3)}$. This motivates the following notion.

Definition 2.5. *For $\varepsilon > 0$ we say a reduced hypergraph $\mathcal{A}$ is* ε-wicked *if it is $(\frac{1}{3} + \varepsilon, \boldsymbol{\wedge})$-dense and fails to support a $K_5^{(3)}$.*

Proposition 2.4 asserts that wicked reduced hypergraphs cannot have too many indices and the proof is divided into two main parts. First we reduce the problem to the case in which the reduced hypergraph $\mathcal{A}$ on some index set I can be *bicoloured*. By this we mean that there is a colouring $\varphi \colon V(\mathcal{A}) \longrightarrow \{\text{red}, \text{blue}\}$ of the vertices such that for every $ij \in I^{(2)}$ we have

$$\varphi^{-1}(\text{red}) \cap \mathcal{P}^{ij} \neq \varnothing \qquad \text{and} \qquad \varphi^{-1}(\text{blue}) \cap \mathcal{P}^{ij} \neq \varnothing \tag{2.1}$$

and there are no hyperedges in $\mathcal{A}$ with all three vertices of the same colour. Given such a colouring φ, we define the *minimum monochromatic codegree density of $\mathcal{A}$ and φ* by

$$\tau_2(\mathcal{A}, \varphi) = \min_{ijk \in I^{(3)}} \min\Big\{\frac{d(P^{ij}, P^{ik})}{|\mathcal{P}^{jk}|} \colon P^{ij} \in \mathcal{P}^{ij},\ P^{ik} \in \mathcal{P}^{ik},\ \text{and } \varphi(P^{ij}) = \varphi(P^{ik})\Big\}\,. \tag{2.2}$$

The following proposition reduces Proposition 2.4 to bicoloured reduced hypergraphs.

Proposition 2.6. *Given $\varepsilon > 0$ and $t \in \mathbb{N}$, let $\mathcal{A}$ be a sufficiently large ε-wicked reduced hypergraph. There exist a reduced hypergraph $\mathcal{A}_\star$ with index set of size at least t not supporting a $K_5^{(3)}$ and a bicolouring φ of $\mathcal{A}_\star$ such that $\tau_2(\mathcal{A}_\star, \varphi) \geqslant \frac{1}{3} + \frac{\varepsilon}{8}$.*

For the proof of Proposition 2.6 we mainly analyse holes in wicked reduced hypergraphs, i.e., subsets of vertices inducing very few edges. It turns out that we can find two "large" but almost disjoint holes such that most edges with two vertices in one of the holes have their third vertex in the other hole. This configuration can then can be used to define an auxiliary reduced hypergraph $\mathcal{A}_\star$ admitting an appropriate colouring φ (see Section 4).

The next proposition completes the proof of Proposition 2.4 by contradicting the conclusion of Proposition 2.6, thus showing that large wicked hypergraphs indeed do not exist.

Proposition 2.7. *For every $\varepsilon > 0$ every sufficiently large bicoloured reduced hypergraph $\mathcal{A}$ with $\tau_2(\mathcal{A}, \varphi) \geqslant \frac{1}{3} + \varepsilon$ supports a $K_5^{(3)}$.*

The proof of Proposition 2.7 is deferred to Section 5.

§3. Preliminaries

In this section we introduce some necessary definitions and properties for reduced hypergraphs.

3.1. **Transversals and cherries.** We start with the following notion for reduced hypergraphs $\mathcal{A}$ with index set I. For $J \subseteq I$ we refer to a set of vertices $\mathcal{Q}(J) = \{Q^{ij} : ij \in J^{(2)}\}$ with $Q^{ij} \in \mathcal{P}^{ij}$ for all $ij \in J^{(2)}$ as a *J-transversal*. Similarly, for two disjoint subsets of indices $K, L \subseteq I$ we say that $\mathcal{Q}(K, L) = \{Q^{k\ell} : (k, \ell) \in K \times L\}$ is a *(K, L)-transversal* when $Q^{k\ell} \in \mathcal{P}^{k\ell}$ for all $(k, \ell) \in K \times L$. Transversals will always be denoted by calligraphic capital letters and the vertices they contain are denoted by the corresponding Roman capital letters (equipped with a pair indices as superscript).

For subsets $J_\star \subseteq J$, $K_\star \subseteq K$, and $L_\star \subseteq L$ we refer to the transversals $\mathcal{Q}(J_\star) \subseteq \mathcal{Q}(J)$ and $\mathcal{Q}(K_\star, L_\star) \subseteq \mathcal{Q}(K, L)$ (defined in the obvious way) as *restricted transversals*. Whenever the sets J, K, $L \subseteq I$ are clear from the context, we may omit them and write *transversal* to refer to J-transversals or to (K, L)-transversals.

Let us recall that we are often assuming implicitly that our index sets are accompanied by a distinguished linear order denoted by $<$. Since we are working with $\curlywedge$-dense reduced hypergraphs (see Definition 2.2), pairs of vertices sharing one index will play an important rôle. More precisely, given indices $ijk \in I^{(3)}$ with $i < j < k$ and given vertices $P^{ij} \in \mathcal{P}^{ij}$, $P^{ik} \in \mathcal{P}^{ik}$, and $P^{jk} \in \mathcal{P}^{jk}$ we say that the ordered pair (P^{ij}, P^{ik}) is a *left cherry*, the ordered pair (P^{ik}, P^{jk}) is a *right cherry*, and the ordered pair (P^{ij}, P^{jk}) is a *middle cherry*. Often we refer to them simply as *cherries*.

For indices $ijk \in I^{(3)}$ and a set of left cherries $\mathscr{L}^{ijk} \subseteq \mathcal{P}^{ij} \times \mathcal{P}^{ik}$ we say a transversal $\mathcal{Q}$ *avoids* $\mathscr{L}^{ijk}$ if $(Q^{ij}, Q^{ik}) \notin \mathscr{L}^{ijk}$ for $Q^{ij}, Q^{ik} \in \mathcal{Q}$. Furthermore, we say $\mathcal{Q}$ avoids a set of left cherries $\mathscr{L} = \bigcup_{ijk \in I^{(3)}} \mathscr{L}^{ijk}$, if it avoids $\mathscr{L}^{ijk}$ for every $ijk \in I^{(3)}$. Similarly, $\mathcal{Q}$ avoids a set of right cherries $\mathscr{R}^{ijk} \subseteq \mathcal{P}^{ik} \times \mathcal{P}^{jk}$ if $(Q^{ik}, Q^{jk}) \notin \mathscr{R}^{ijk}$, and $\mathcal{Q}$ avoids a set of right cherries $\mathscr{R} = \bigcup_{ijk \in I^{(3)}} \mathscr{R}^{ijk}$ if it avoids each $\mathscr{R}^{ijk}$. Note that these definitions apply both to J-transversals and to (K, L)-transversals.

3.2. **Inhabited transversals in weakly dense reduced hypergraphs.** We shall utilise a key result from [13] on $\therefore$-dense hypergraphs. Roughly speaking, this notion concerns hypergraphs which have a uniform edge distribution on large sets of vertices. However, here we restrict ourselves to the corresponding concepts for reduced hypergraphs arising after an application of the hypergraph regularity lemma (see, e.g., [10, 13] for more details).

Definition 3.1. *Let $\mu > 0$ and let $\mathcal{A}$ be a reduced hypergraph on an index set I. We say that $\mathcal{A}$ is* $(\mu, \therefore)$-dense, *if for every $ijk \in I^{(3)}$ we have*

$$e(\mathcal{A}^{ijk}) \geqslant \mu\, |\mathcal{P}^{ij}||\mathcal{P}^{ik}||\mathcal{P}^{jk}|\,. \tag{3.1}$$

Further, for disjoint subsets of indices $K, L, M \subseteq I$ we say that $\mathcal{A}$ is $(\mu, \therefore)$-tridense *on K, L, M, if (3.1) holds for every triple (i, j, k) in $K \times L \times M$.*

Note that by definition every $(d, \wedge)$-dense reduced hypergraph is also $(d, \therefore)$-dense. The following result from [13, Lemma 3.1] states the existence of transversals containing edges in $\therefore$-dense reduced hypergraphs.

Theorem 3.2. *Let $t \in \mathbb{N}$, $\mu > 0$, and let $\mathcal{A}$ be a $(\mu, \therefore)$-dense reduced hypergraph on a sufficiently large index set I. There exist a set $I_\star \subseteq I$ of size t and three transversals $\mathcal{Q}(I_\star)$, $\mathcal{R}(I_\star)$, and $\mathcal{S}(I_\star)$ such that $Q^{ij} R^{ik} S^{jk} \in E(\mathcal{A})$ for all $i < j < k$ in $I_\star$.* $\square$

Triples of transversals satisfying the conclusion of Theorem 3.2 will play an important rôle here and this motivates the following definition.

Definition 3.3 (inhabited triple of transversals)**.** *Given a reduced hypergraph $\mathcal{A}$ with index set I, we say a triple of transversals $\mathcal{Q}(J)\mathcal{R}(J)\mathcal{S}(J)$ for some $J \subseteq I$ is* inhabited *if for all $i < j < k$ in J we have $Q^{ij} R^{ik} S^{jk} \in E(\mathcal{A})$.*

Similarly, for pairwise disjoint sets of indices K, L, $M \subseteq I$, we say a triple of transversals $\mathcal{Q}(K, L)\mathcal{R}(K, M)\mathcal{S}(L, M)$ is inhabited *if for every $k \in K$, $\ell \in L$, and $m \in M$ we have $Q^{k\ell} R^{km} S^{\ell m} \in E(\mathcal{A})$.*

We will also need a version of Theorem 3.2 in which the resulting transversals avoid given sets of forbidden cherries.

Lemma 3.4. *For all $t \in \mathbb{N}$ and $\mu > 0$ there is $\mu' > 0$ such that the following holds. Let $\mathcal{A}$ be a $(\mu, \therefore)$-dense reduced hypergraph on a sufficiently large index set I and for all $i < j < k$ in I let $\mathscr{L}^{ijk} \subseteq \mathcal{P}^{ij} \times \mathcal{P}^{ik}$ and $\mathscr{R}^{ijk} \subseteq \mathcal{P}^{ik} \times \mathcal{P}^{jk}$ be sets of left and right cherries satisfying*

$$|\mathscr{L}^{ijk}| \leqslant \mu' |\mathcal{P}^{ij}||\mathcal{P}^{ik}| \qquad \text{and} \qquad |\mathscr{R}^{ijk}| \leqslant \mu' |\mathcal{P}^{ik}||\mathcal{P}^{jk}|\,.$$

There exist a set $I_\star \subseteq I$ of size t and an inhabited triple of transversals $\mathcal{Q}(I_\star)\mathcal{R}(I_\star)\mathcal{S}(I_\star)$ avoiding the cherries $\mathscr{L}^{ijk}$ and $\mathscr{R}^{ijk}$ for every $ijk \in I_\star^{(3)}$.

For the proof of Lemma 3.4 we will consider random preimages of reduced hypergraphs.

Definition 3.5 (random preimage). *Given a reduced hypergraph $\mathcal{A}$ with index set I and vertex classes $\mathcal{P}^{ij}$ for $ij \in I^{(2)}$, and given an integer $\ell \geqslant 1$, we fix $\binom{|I|}{2}$ mutually disjoint sets $\mathcal{P}^{ij}_\bullet$ of size ℓ and consider the uniform probability space $\mathfrak{A}(\mathcal{A}, \ell)$ of all mappings h from $\bigcup_{ij\in I^{(2)}} \mathcal{P}^{ij}_\bullet$ to $\bigcup_{ij\in I^{(2)}} \mathcal{P}^{ij}$ satisfying*

$$h(\mathcal{P}^{ij}_\bullet) \subseteq \mathcal{P}^{ij}$$

for every $ij \in I^{(2)}$.

With each such map h we associate a reduced hypergraph $\mathcal{A}_h$ with index set I and vertex classes $\mathcal{P}^{ij}_\bullet$ for $ij \in I^{(2)}$ whose edges are defined by

$$P^{ij}_\bullet P^{ik}_\bullet P^{jk}_\bullet \in E(\mathcal{A}^{ijk}_h) \quad\Longleftrightarrow\quad h(P^{ij}_\bullet)h(P^{ik}_\bullet)h(P^{jk}_\bullet) \in E(\mathcal{A}^{ijk})$$

for all $ijk \in I^{(3)}$ and all $P^{ij}_\bullet \in \mathcal{P}^{ij}_\bullet$, $P^{ik}_\bullet \in \mathcal{P}^{ik}_\bullet$, and $P^{jk}_\bullet \in \mathcal{P}^{jk}_\bullet$.

Notice that in this situation h is a hypergraph homomorphism from $\mathcal{A}_h$ to $\mathcal{A}$. Below we pass to such a random preimage $\mathcal{A}_h$ of $\mathcal{A}$ for sufficiently large ℓ, which will allow us to deduce Lemma 3.4 for $\mathcal{A}$ by applying Theorem 3.2 to $\mathcal{A}_h$.

Proof of Lemma 3.4. Given $t \in \mathbb{N}$ and $\mu > 0$, let t_1 be sufficiently large for an application of Theorem 3.2 with t and $\frac{\mu}{2}$ in place of t and μ. Further, we fix an integer ℓ and $\mu' > 0$ to satisfy the hierarchy

$$\mu, t_1^{-1} \gg \ell^{-1} \gg \mu'.$$

Finally, let $\mathcal{A}$ be a reduced hypergraph as in the statement of Lemma 3.4. We may assume that its index set I is of size t_1.

Similar as in the proof of [10, Lemma 4.2] we consider the probability space $\mathfrak{A}(\mathcal{A}, \ell)$ from Definition 3.5 and we shall prove that with high probability the associated reduced hypergraph $\mathcal{A}_h$ is $(\frac{\mu}{2}, \therefore)$-dense and no cherry has its image in the sets $\mathscr{L}^{ijk}$ or $\mathscr{R}^{ijk}$.

For every constituent $\mathcal{A}^{ijk}_h$ the random variable $e(\mathcal{A}^{ijk}_h)$ satisfies $\mathbb{E}[e(\mathcal{A}^{ijk}_h)] \geqslant \mu\ell^3$ and by Azuma's inequality (see, e.g., [8, Corollary 2.27]) we obtain

$$\mathbb{P}\big(\mathcal{A}_h \text{ is not } (\tfrac{\mu}{2}, \therefore)\text{-dense}\big) \leqslant \sum_{ijk\in I^{(3)}} \mathbb{P}\big(e(\mathcal{A}^{ijk}_h) < \tfrac{\mu}{2}\ell^3\big) \leqslant \binom{t_1}{3} \exp\big(-\tfrac{\mu^2\ell}{24}\big)\,.$$

Moreover, since $\mathscr{L}^{ijk} \leqslant \mu'|\mathcal{P}^{ij}||\mathcal{P}^{ik}|$, the probability that the image of some cherry lies in those sets is bounded by

$$\sum_{ijk\in I^{(3)}} \mathbb{P}\Big(h(P^{ij}_\bullet)h(P^{ik}_\bullet) \in \mathscr{L}^{ijk} \text{ for some } P^{ij}_\bullet P^{ik}_\bullet \in \mathcal{P}^{ij}_\bullet \times \mathcal{P}^{ik}_\bullet\Big) \leqslant \binom{t_1}{3}\mu'\ell^2\,.$$

The same inequality holds for the sets $\mathscr{R}^{ijk}$ and our choice of parameters ensures

$$\binom{t_1}{3} \exp\big(-\tfrac{\mu^2\ell}{24}\big) + 2\binom{t_1}{3}\mu'\ell^2 < 1\,.$$

Therefore, we can fix an h such that $\mathcal{A}_h$ is $(\frac{\mu}{2}, \therefore)$-dense and no cherry has its image in the sets $\mathscr{L}^{ijk}$ or $\mathscr{R}^{ijk}$.

Applying Theorem 3.2 to $\mathcal{A}_h$ yields a set $I_\star \subseteq I$ of size t and a triple of transversals $\mathcal{Q}_h(I_\star)\mathcal{R}_h(I_\star)\mathcal{S}_h(I_\star)$ inhabited in $\mathcal{A}_h$. It is easy to see that the transversals

$$\mathcal{Q}(I_\star) = h\big(\mathcal{Q}(I_\star)\big)\,, \quad \mathcal{R}(I_\star) = h\big(\mathcal{R}(I_\star)\big)\,, \quad \text{and} \quad \mathcal{S}(I_\star) = h\big(\mathcal{S}(I_\star)\big)$$

are as required. □

3.3. **Partite versions.** We will also need a slightly more involved variant of Theorem 3.2, which guarantees the existence of inhabited triples of transversals in the intersection of multiple $\therefore$-tridense reduced subhypergraphs.

Lemma 3.6. *For all t, $r \in \mathbb{N}$, $\mu > 0$ there is some $s \in \mathbb{N}$ such that the following is true. Let $\mathcal{A}$ be a reduced hypergraph on index set I. Suppose that we have*

- *(a) disjoint subsets of indices $K, L, M \subseteq I$ each of size s,*
- *(b) sets $X_1, \dots, X_r$ of size s, and*
- *(c) for every r-tuple $\vec{x} \in \prod_{i\in[r]} X_i$ a $(\mu, \therefore)$-tridense subhypergraph $\mathcal{A}_{\vec{x}} \subseteq \mathcal{A}$ on K, L, M.*

Then, there are

- *(i) subsets $K_\star \subseteq K, L_\star \subseteq L, M_\star \subseteq M$ of size t,*
- *(ii) subsets $Y_i \subseteq X_i$ of size t for every $i \in [r]$, and*
- *(iii) a triple of transversals $\mathcal{Q}(K_\star, L_\star)\mathcal{R}(K_\star, M_\star)\mathcal{S}(L_\star, M_\star)$, which is inhabited in $\mathcal{A}_{\vec{y}}$ for every $\vec{y} \in \prod_{i\in[r]} Y_i$.*

The proof of Lemma 3.6 relies on three successive applications of the following auxiliary lemma.

Lemma 3.7. *For all t, $r \in \mathbb{N}$, $\mu > 0$ there is some $s \in \mathbb{N}$ such that the following is true. Let $\mathcal{A}$ be a reduced hypergraph on index set I. Suppose that we have*

- *(a) disjoint subsets of indices $K, L \subseteq I$ each of size s,*
- *(b) sets $X_1, \dots, X_r$ of size s, and*
- *(c) for every r-tuple $\vec{x} \in \prod_{i\in[r]} X_i$, every $k \in K$, and every $\ell \in L$ a subset $\mathcal{P}^{kl}_{\vec{x}} \subseteq \mathcal{P}^{kl}$ of size at least $\mu|\mathcal{P}^{k\ell}|$.*

Then, there are

- *(i) subsets $K' \subseteq K, L' \subseteq L$ of size t,*
- *(ii) subsets $X'_i \subseteq X_i$ of size t for every $i \in [r]$, and*
- *(iii) a transversal $\mathcal{Q}(K', L')$ such that for every $\vec{x} \in \prod_{i\in[r]} X'_i$ and every $(k, \ell) \in K' \times L'$ we have that $Q^{k\ell} \in \mathcal{P}^{k\ell}_{\vec{x}}$.*

Proof. Given t, $r \in \mathbb{N}, \mu > 0$ we fix an integer s such that

$$t, r, \mu^{-1} \ll s\,. \tag{3.2}$$

Let $\mathcal{A}$ be a reduced hypergraph as in the statement of the lemma and further let $K' \subseteq K$, and $L' \subseteq L$ be arbitrary subsets of size t.

For every (K', L')-transversal $\mathcal{Q}$ we consider the set

$$\mathfrak{x}(\mathcal{Q}) = \left\{ \vec{x} \in \prod_{i \in [r]} X_i \colon Q^{k\ell} \in \mathcal{P}^{k\ell}_{\vec{x}} \text{ for all } (k, \ell) \in K' \times L' \right\}.$$

Summing over all (K', L')-transversals $\mathcal{Q}$ assumption *(c)* yields

$$\sum_{\mathcal{Q}} |\mathfrak{x}(\mathcal{Q})| = \sum_{\vec{x} \in \prod_{i \in [r]} X_i} \prod_{(k,\ell) \in K' \times L'} \left|\mathcal{P}^{k\ell}_{\vec{x}}\right| \geqslant \mu^{t^2} \prod_{(k,\ell) \in K' \times L'} \left|\mathcal{P}^{k\ell}\right| \prod_{i \in [r]} |X_i|.$$

Hence, we can fix a (K', L')-transversal $\mathcal{Q}$ such that

$$|\mathfrak{x}(\mathcal{Q})| \geqslant \mu^{t^2} \prod_{i \in [r]} |X_i|.$$

We may view $\mathfrak{x}(\mathcal{Q})$ as an r-partite r-uniform hypergraph of density at least μ^{t^2} on vertex classes of size s. Consequently, a result of Erdős [3] combined with the hierarchy (3.2) yields subsets $X'_i \subseteq X_i$ of size t for every $i \in [r]$ such that

$$\prod_{i \in [r]} X'_i \subseteq \mathfrak{x}(\mathcal{Q}),$$

which concludes the proof of Lemma 3.7. □

Next we derive Lemma 3.6.

Proof of Lemma 3.6. Given $t, r \in \mathbb{N}, \mu > 0$ we fix integers s, s', and s'' such that

$$t, r, \mu^{-1} \ll s'' \ll s' \ll s$$

and let $\mathcal{A}$ be a reduced hypergraph as in the statement of the lemma. We will prove the lemma by applying Lemma 3.7 three times, once for every pair from K, L, and M.

First step. For every $k \in K$, $\ell \in L$, $m \in M$, and every $\vec{x} \in \prod_{i \in [r]} X_i$ we consider the set

$$\mathcal{P}^{k\ell}_{(\vec{x},m)} = \left\{ P^{k\ell} \in \mathcal{P}^{k\ell} \colon \left|N_{\mathcal{A}^{k\ell m}_{\vec{x}}}(P^{k\ell})\right| \geqslant \tfrac{\mu}{2} |\mathcal{P}^{km}||\mathcal{P}^{\ell m}| \right\}.$$

Since $\mathcal{A}_{\vec{x}}$ is $(\mu, \therefore)$-tridense, we have

$$e(A^{k\ell m}_{\vec{x}}) \geqslant \mu |\mathcal{P}^{k\ell}||\mathcal{P}^{km}||\mathcal{P}^{\ell m}|,$$

and a standard counting argument implies

$$\left|\mathcal{P}^{k\ell}_{(\vec{x},m)}\right| \geqslant \tfrac{\mu}{2} \left|\mathcal{P}^{k\ell}\right|.$$

Lemma 3.7 applied with s', $r+1$, and $\frac{\mu}{2}$ in place of t, r, and μ and with $X_{r+1} = M$ yields s'-element subsets $K' \subseteq K$, $L' \subseteq L$, $M' \subseteq M$, and $X'_i \subseteq X_i$ for every $i \in [r]$ and a transversal $\mathcal{Q}(K', L')$ such that for every $(\vec{x}, m) \in \prod_{i \in [r]} X'_i \times M'$ and every $(k, \ell) \in K' \times L'$ we have that $Q^{k\ell} \in \mathcal{P}^{k\ell}_{(\vec{x},m)}$.

Second Step. Next we consider for every $k \in K'$, $\ell \in L'$, $m \in M'$, and every $\vec{x} \in \prod_{i\in[r]} X'_i$ the set

$$\mathcal{P}^{km}_{(\vec{x},\ell)} = \left\{ P^{km} \in \mathcal{P}^{km} \colon \left| N_{\mathcal{A}^{k\ell m}_{\vec{x}}}(Q^{k\ell}, P^{km}) \right| \geqslant \tfrac{\mu}{4} |\mathcal{P}^{\ell m}| \right\}.$$

By our choice of the transversal $\mathcal{Q}(K', L')$ we have

$$|N_{\mathcal{A}^{k\ell m}_{\vec{x}}}(Q^{k\ell})| \geqslant \tfrac{\mu}{2} |\mathcal{P}^{km}||\mathcal{P}^{\ell m}|$$

and, as before, this implies

$$|\mathcal{P}^{km}_{(\vec{x},\ell)}| \geqslant \tfrac{\mu}{4} |\mathcal{P}^{km}|.$$

Again, we apply Lemma 3.7, now with s'', $r+1$, and $\frac{\mu}{4}$ in place of t, r, and μ and with $X'_{r+1} = L'$, to reach s''-element subsets $K'' \subseteq K'$, $L'' \subseteq L'$, $M'' \subseteq M'$, and $X''_i \subseteq X'_i$, for every $i \in [r]$ and a transversal $\mathcal{R}(K'', M'')$ such that for every $(\vec{x}, \ell) \in \prod_{i\in[r]} X''_i \times L''$ and every $(k, m) \in K'' \times M''$ we have $R^{km} \in \mathcal{P}^{km}_{(\vec{x},\ell)}$.

Third step. Last, we consider for every $\ell \in L''$, $k \in K''$, $m \in M''$, and every $\vec{x} \in \prod_{i\in[r]} X''_i$ the set

$$\mathcal{P}^{\ell m}_{(\vec{x},k)} = N_{\mathcal{A}^{k\ell m}_{\vec{x}}}(Q^{k\ell}, R^{km}).$$

By our choice of the transversals $\mathcal{Q}(K'', L'')$ and $\mathcal{R}(K'', M'')$ we have $|\mathcal{P}^{\ell m}_{(\vec{x},k)}| \geqslant \frac{\mu}{4} |\mathcal{P}^{\ell m}|$. The final application of Lemma 3.7, with t, $r+1$, and $\frac{\mu}{4}$ in place of t, r, and μ, yields t-sized subsets $K_\star \subseteq K''$, $L_\star \subseteq L''$, $M_\star \subseteq M''$, and $Y_i \subseteq X''_i$, for every $i \in [r]$, and a transversal $\mathcal{S}(L_\star, M_\star)$ such that for every $\vec{y} \in \prod_{i\in[r]} Y_i$ and every $(k, \ell, m) \in K_\star \times L_\star \times M_\star$ we have that $Q^{k\ell} R^{km} S^{\ell m} \in E(\mathcal{A}_{\vec{y}})$. In other words, the triple of transversals $\mathcal{QRS}$ is inhabited in every $\mathcal{A}_{\vec{y}}$ with $\vec{y} \in \prod_{i\in[r]} Y_i$. $\square$

§4. Bicolouring wicked reduced hypergraphs

4.1. **Plan.** This entire section is devoted to the proof of Proposition 2.6. As the argument is quite long, we would like to commence with a brief outline of our strategy.

4.1.1. *Naïve ideas.* In an attempt to keep this account sufficiently digestible we will systematically oversimplify and most claims below will later turn out to be true in a metaphorical sense only.

It might be helpful to know, what the proof of Proposition 2.6 does, when the given ε-wicked reduced hypergraph $\mathcal{A}$ itself possesses a bicolouring φ of the vertices (which might be “unknown“ to us) and to contrast this situation with the general case.

What can immediately be seen is that in the bicoloured case $\mathcal{A}$ contains many holes, by which we mean that there are many independent sets $\Phi \subseteq V(\mathcal{A})$ such that for every pair of indices ij we have $|\mathcal{P}^{ij} \cap \Phi| \geqslant (1/3 + \varepsilon)|\mathcal{P}^{ij}|$. Indeed, there are “red holes” consisting of red vertices only and, similarly, there are “blue holes”. For the sake of discussion we will pretend that these are the only holes, i.e., that each hole is either red or blue.

In the general case, it might not be clear on first sight that any holes exist, but based on the assumption that $\mathcal{A}$ fails to support a $K_5^{(3)}$ one can establish that they do. As a matter of fact, there is a fairly flexible method to construct holes and thus one should think of the set $\mathfrak{H}$ of all holes in $\mathcal{A}$ as having a possibly intricate structure. There are three main lemmata in our analysis of $\mathfrak{H}$:

- the transitivity lemma;
- the union lemma;
- and the density increment lemma.

Let us briefly summarise the content of these three statements.

I. Returning to the bicoloured case, "being of the same colour" is an obvious equivalence relation on $\mathfrak{H}$, which has two equivalence classes. Moreover, if φ and thus the colouring of the holes is unknown, this equivalence relation is definable by saying that two holes are equivalent if and only if they intersect each other (substantially) on every vertex class $\mathcal{P}^{ij}$. When $\mathcal{A}$ is arbitrary, the relation of intersecting each other in this sense is clearly reflexive and symmetric. The aforementioned *transitivity lemma* ensures that this relation is transitive as well; its proof requires some effort. One can also show that $\mathfrak{H}$ always consists of exactly two equivalence classes.

II. In the bicoloured case, the union of two red holes is again a red hole and, in fact, the class of red vertices is definable as the union of all red holes. It turns out that in the general case one can prove the union of two equivalent holes to be a hole as well, and this is what the *union lemma* asserts.

III. Iterative applications of the union lemma yield two maximal holes, namely the unions of the two equivalence classes. In the bicoloured case every vertex belongs to one of these maximal holes, but this is not necessary for the proof of Proposition 2.6 to go through. All that matters is that for two appropriate holes

$$\text{most edges with two vertices in one hole have their third vertex in the other hole.} \quad (*)$$

The density increment lemma states that if two holes violate $(*)$, then there are two other holes covering more space. Thus iterative applications of this lemma show that the maximal holes satisfy $(*)$.

Notice that if we managed to arrive at two holes satisfying $(*)$, then the proof of Proposition 2.6 could be completed by deleting the vertices not belonging to them.

4.1.2. *A more realistic picture.* Let us now point to two deficiencies of the foregoing outline. First, we will never show that the given reduced hypergraph $\mathcal{A}$ contains a hole containing no edges at all. All we need and prove is that there are large sets inducing very few edges in $\mathcal{A}$ and thus there will be parameters μ, ν, etc. quantifying how accurate our holes are. Second, each step of the argument is accompanied by a significant loss of the relevant part of the index set. Thus the number of times we apply our key lemmata needs to be bounded by a function of ε and, therefore, we will never reach holes that are maximal in the absolute sense. All that can realistically be said is that there are two holes which cannot be enlarged by a substantial amount, and for this reason we adopt the somewhat indirect density increment formulation of the third main lemma.

4.1.3. *Organisation.* In §4.2 and §4.3 we deal with general properties of $\boldsymbol{\wedge}$-dense reduced hypergraphs not supporting a $K_5^{(3)}$, including the existence of holes. The main result of §4.4 is the transitivity lemma, a precise version of which will be stated as Lemma 4.10. Next, the union lemma is obtained in §4.5 (see Lemma 4.13). The proof of the density increment lemma (Lemma 4.17) requires some preparations provided in §4.6, while the proof itself is given in §4.7. Finally, we argue in §4.8 that despite the approximate nature of the arguments provided so far the proof of Proposition 2.6 can be completed by taking a random preimage.

4.2. **Holes and links in reduced hypergraphs.** Given a reduced hypergraph $\mathcal{A}$ with index set I, a natural definition of a *hole* across a subset of indices $J \subseteq I$ and subsets of vertices $\Phi^{ij} \subseteq \mathcal{P}^{ij}$ for $ij \in J^{(2)}$ would maybe require that for every $ijk \in J^{(3)}$ the sets Φ^{ij}, Φ^{ik}, Φ^{jk} span no hyperedges in $\mathcal{A}^{ijk}$. However, this notion is too restrictive for our analysis and we shall only require that these sets induce hypergraphs of low density.

Definition 4.1. *Given a reduced hypergraph $\mathcal{A}$ and a subset of indices $J \subseteq I$ we say that a subset of vertices $\Phi \subseteq V(\mathcal{A})$ is a μ-*hole *on J if $\Phi^{ij} = \Phi \cap \mathcal{P}^{ij}$ is nonempty for all $ij \in J^{(2)}$ and*

$$e(\Phi^{ij}, \Phi^{ik}, \Phi^{jk}) \leqslant \mu|\mathcal{P}^{ij}||\mathcal{P}^{ik}||\mathcal{P}^{jk}|$$

for every $ijk \in J^{(3)}$.

The size *of the hole is $|J|$ and the smallest $\varsigma > 0$ such that $|\Phi^{ij}| \geqslant \varsigma|\mathcal{P}^{ij}|$ for every $ij \in J^{(2)}$ is called the* width *of the hole. We refer to μ-holes with width at least ς as (μ, ς)-holes.*

Roughly speaking, for the proof of Proposition 2.6 we shall find two almost disjoint holes with widths bigger than 1/3 on a large set of indices in a wicked reduced hypergraph.

Holes may induce a few hyperedges. However, cherries that are contained in too many such hyperedges are considered to be exceptional. This leads to the following definition.

Definition 4.2. *Given a μ-hole Φ on J, $\varepsilon > 0$, and ijk in $J^{(3)}$ a cherry $(P^{ij}, P^{ik}) \in \Phi^{ij} \times \Phi^{ik}$ is ε-*exceptional *if*

$$\left|N(P^{ij}, P^{ik}) \cap \Phi^{jk}\right| \geqslant \varepsilon\left|\mathcal{P}^{jk}\right|.$$

For indices $i < j < k$ in J we denote by

$$\mathscr{L}^{ijk}(\Phi, \varepsilon) \subseteq \mathcal{P}^{ij} \times \mathcal{P}^{ik}\,, \quad \mathscr{M}^{ijk}(\Phi, \varepsilon) \subseteq \mathcal{P}^{ij} \times \mathcal{P}^{jk}\,, \quad \text{and} \quad \mathscr{R}^{ijk}(\Phi, \varepsilon) \subseteq \mathcal{P}^{ik} \times \mathcal{P}^{jk}\,,$$

the ε-exceptional left, middle, and right cherries and we set

$$\mathscr{L}(\Phi,\varepsilon)=\dot{\bigcup_{i<j<k}}\mathscr{L}^{ijk}(\Phi,\varepsilon)\,,\quad \mathscr{M}(\Phi,\varepsilon)=\dot{\bigcup_{i<j<k}}\mathscr{M}^{ijk}(\Phi,\varepsilon)\,,\quad \textit{and}\ \mathscr{R}(\Phi,\varepsilon)=\dot{\bigcup_{i<j<k}}\mathscr{R}^{ijk}(\Phi,\varepsilon)\,.$$

It is easy to see that holes can only contain few exceptional cherries. More precisely, for every μ-hole Φ on J and every $\varepsilon>0$ we have for all $i<j<k$ in J

$$\varepsilon\,|\mathcal{P}^{jk}||\mathscr{L}^{ijk}(\Phi,\varepsilon)|\leqslant e(\Phi^{ij},\Phi^{ik},\Phi^{jk})\leqslant\mu\,|\mathcal{P}^{ij}||\mathcal{P}^{ik}||\mathcal{P}^{jk}|$$

and the same reasoning applies to $\mathscr{R}$ and $\mathscr{M}$. This shows

$$\big|\mathscr{L}^{ijk}(\Phi,\varepsilon)\big|\leqslant\frac{\mu}{\varepsilon}\big|\mathcal{P}^{ij}\big|\big|\mathcal{P}^{ik}\big|\,,\quad \big|\mathscr{M}^{ijk}(\Phi,\varepsilon)\big|\leqslant\frac{\mu}{\varepsilon}\big|\mathcal{P}^{ij}\big|\big|\mathcal{P}^{jk}\big|\,,$$
$$\text{and}\quad \big|\mathscr{R}^{ijk}(\Phi,\varepsilon)\big|\leqslant\frac{\mu}{\varepsilon}\big|\mathcal{P}^{ik}\big|\big|\mathcal{P}^{jk}\big|\,. \tag{4.1}$$

Often we consider holes Φ arising from neighbourhoods $N(P^{ik},P^{jk})$, i.e., for appropriately chosen $P^{ik}\in\mathcal{P}^{ik}$ and $P^{jk}\in\mathcal{P}^{jk}$ we set $\Phi^{ij}=N(P^{ik},P^{jk})$. Note that in $(d,\boldsymbol{\wedge})$-dense reduced hypergraphs, holes obtained in this way will automatically have width at least d.

Given a (K,L)-transversal $\mathcal{Q}$, a subset $K_\star\subseteq K$, and an index $\ell\in L$ we define the $\mathcal{Q}$*-link of ℓ on $K_\star$* by

$$\Lambda(\mathcal{Q},K_\star,\ell)=\dot{\bigcup_{kk'\in K_\star^{(2)}}}N(Q^{k\ell},Q^{k'\ell})\,.$$

The following lemma asserts that in $\boldsymbol{\wedge}$-dense reduced hypergraphs that do not support $K_5^{(3)}$ the $\mathcal{Q}$-links contain large holes.

Lemma 4.3. *Let $t\in\mathbb{N}$, μ, $d>0$, let $\mathcal{A}$ be a $(d,\boldsymbol{\wedge})$-dense reduced hypergraph with index set I that does not support a $K_5^{(3)}$, and for sufficiently large disjoint subsets of indices $K,L\subseteq I$ let $\mathcal{Q}$ be a (K,L)-transversal.*

Then there exist $K_\star\subseteq K$ and $L_\star\subseteq L$ of size t such that for every $\ell\in L_\star$ the link $\Lambda(\mathcal{Q},K_\star,\ell)$ is a (μ,d)-hole .

Proof. Let $q=\binom{\lceil\mu^{-1}\rceil}{2}$ and set $\Upsilon_\ell^{kk'}=N(Q^{k\ell},Q^{k'\ell})$ for all $kk'\in K^{(2)}$, $\ell\in L$, and, similarly, $\Upsilon_k^{\ell\ell'}=N(Q^{k\ell},Q^{k\ell'})$ for all $k\in K$, $\ell\ell'\in L^{(2)}$. Consider an auxiliary 2-colouring of the pairs $(kk'k'',\ell)\in K^{(3)}\times L$ depending on whether

$$e\big(\Upsilon_\ell^{kk'},\Upsilon_\ell^{kk''},\Upsilon_\ell^{k'k''}\big)>\mu|\mathcal{P}^{kk'}||\mathcal{P}^{kk''}||\mathcal{P}^{k'k''}| \tag{4.2}$$

holds or not. Since K and L are sufficiently large, the product Ramsey theorem (see, e.g., [7, Theorem 5.1.5]) yields a set $K_1\subseteq K$ with $|K_1|\geqslant\max\{3d^{-q},t\}$ and a set $L_1\subseteq L$ with $|L_1|\geqslant\max\{\lceil\mu^{-1}\rceil,t\}$ such that all pairs $(kk'k'',\ell)\in K_1^{(3)}\times L_1$ agree whether (4.2) holds or not. In fact, if (4.2) fails on $K_1^{(3)}\times L_1$, then arbitrary t-element subsets $K_\star\subseteq K_1$ and $L_\star\subseteq L_1$ have the desired property. Consequently, we may assume that (4.2) holds on $K_1^{(3)}\times L_1$. We shall show that this implies $\mathcal{A}$ to support a $K_5^{(3)}$.

Let L_2 be a subset of L_1 of size $|L_2| = \lceil \mu^{-1} \rceil$ and consider some $\ell\ell' \in L_2^{(2)}$. Since we have $|\Upsilon_k^{\ell\ell'}| \geqslant d|\mathcal{P}^{\ell\ell'}|$ for every $k \in K_1$, there is a subset $K_2 \subseteq K_1$ of size at least $d|K_1|$ such that

$$\bigcap_{k \in K_2} \Upsilon_k^{\ell\ell'} \neq \varnothing \, .$$

Repeating this argument iteratively $q = \binom{|L_2|}{2}$ times, once for every pair in L_2, we obtain nested subsets $K_1 \supseteq K_2 \supseteq \dots \supseteq K_{q+1}$ such that

$$|K_{q+1}| \geqslant d^q |K_1| \geqslant 3 \quad \text{and} \quad \bigcap_{k \in K_{q+1}} \Upsilon_k^{\ell\ell'} \neq \varnothing \text{ for every } \ell\ell' \in L_2^{(2)} \, .$$

The first statement allows us to fix some $kk'k'' \in K_{q+1}^{(3)}$ and the second one yields for every $\ell\ell' \in L_2^{(2)}$ a fixed vertex $P^{\ell\ell'} \in \mathcal{P}^{\ell\ell'}$ satisfying

$$P^{\ell\ell'} Q^{k\ell} Q^{k\ell'}, \; P^{\ell\ell'} Q^{k'\ell} Q^{k'\ell'}, \; P^{\ell\ell'} Q^{k''\ell} Q^{k''\ell'} \in E(\mathcal{A}) \, . \tag{4.3}$$

We infer from (4.2) and our choice of L_2 that

$$\sum_{\ell \in L_2} e\big(\Upsilon_\ell^{kk'}, \Upsilon_\ell^{kk''}, \Upsilon_\ell^{kk''}\big) > \mu |L_2| |\mathcal{P}^{kk'}| |\mathcal{P}^{kk''}| |\mathcal{P}^{k'k''}| \geqslant |\mathcal{P}^{kk'}| |\mathcal{P}^{kk''}| |\mathcal{P}^{k'k''}| \, .$$

Consequently, there are an edge $R^{kk'} R^{kk''} R^{k'k''} \in E(\mathcal{A}^{kk'k''})$ and two distinct indices $\ell, \ell' \in L_2$ such that both $\lambda \in \{\ell, \ell'\}$ satisfy

$$R^{kk'} Q^{k\lambda} Q^{k'\lambda}, \; R^{kk''} Q^{k\lambda} Q^{k''\lambda}, \; R^{k'k''} Q^{k'\lambda} Q^{k''\lambda} \in E(\mathcal{A}) \, .$$

Together with (4.3) we arrive at the contradiction that $P^{\ell\ell'}$, the six vertices $Q^{\kappa\lambda}$ with $\kappa \in \{k, k', k''\}$ and $\lambda \in \{\ell, \ell'\}$, and the three vertices $R^{kk'}$, $R^{kk''}$, $R^{k'k''}$ support a $K_5^{(3)}$ in $\mathcal{A}$. $\square$

Two consecutive applications of Lemma 4.3 yield the symmetric conclusion that both links $\Lambda(\mathcal{Q}, K_\star, \ell)$ and $\Lambda(\mathcal{Q}, L_\star, k)$ are μ-holes for every $\ell \in L_\star$ and $k \in K_\star$.

Corollary 4.4. *Let $t \in \mathbb{N}$, μ, $d > 0$, let $\mathcal{A}$ be a $(d, \wedge)$-dense reduced hypergraph with index set I that does not support a $K_5^{(3)}$, and for sufficiently large disjoint subsets of indices $K, L \subseteq I$ let $\mathcal{Q}$ be a (K, L)-transversal.*

Then there exist $K_\star \subseteq K$ and $L_\star \subseteq L$ of size t such that for every $\ell \in L_\star$ and for every $k \in K_\star$ the $\mathcal{Q}$-links $\Lambda(\mathcal{Q}, K_\star, \ell)$ and $\Lambda(\mathcal{Q}, L_\star, k)$ are (μ, d)-holes.

Proof. For sufficiently large $t' = t'(t, \mu, d)$ a first application of Lemma 4.3 yields subsets K' and L' of size at least t' such that $\Lambda(\mathcal{Q}, K', \ell)$ is a (μ, d)-hole for every $\ell \in L'$. A second application to the restricted transversal $\mathcal{Q}(K', L')$ (with the rôles of K and L exchanged) then yields subsets $L_\star \subseteq L'$ and $K_\star \subseteq K'$ of size t such that additionally $\Lambda(\mathcal{Q}, L_\star, k)$ is a (μ, d)-hole for every $k \in K_\star$. $\square$

4.3. **Intersecting and disjoint links.** Next we define concepts for pairs of links of having a substantial intersection and of being almost disjoint.

Definition 4.5. *Let $\mathcal{A}$ be a reduced hypergraph with index set I, let $K, L, M \subseteq I$ be pairwise disjoint sets of indices, and let $\mathcal{Q}(K, L)$ and $\mathcal{R}(K, M)$ be transversals.*

For $\ell \in L$ and $m \in M$ we say the links $\Lambda(\mathcal{Q}, K, \ell)$ and $\Lambda(\mathcal{R}, K, m)$ are δ-intersecting *if*

$$\big|N(Q^{k\ell}, Q^{k'\ell}) \cap N(R^{km}, R^{k'm})\big| \geqslant \delta\big|\mathcal{P}^{kk'}\big| \tag{4.4}$$

for all $kk' \in K^{(2)}$. If, on the other hand, (4.4) fails for all $kk' \in K^{(2)}$, then we say $\Lambda(\mathcal{Q}, K, \ell)$ and $\Lambda(\mathcal{R}, K, m)$ are δ-disjoint.

Moreover, we say a pair of transversals $\mathcal{Q}(K, L)\mathcal{R}(K, M)$ has δ-intersecting links *(resp.* δ-disjoint links*) if $\Lambda(\mathcal{Q}, K, \ell)$ and $\Lambda(\mathcal{R}, K, m)$ are δ-intersecting (resp. δ-disjoint) for every $\ell \in L$ and $m \in M$.*

We remark that the notions of being δ-intersecting and δ-disjoint do not complement each other. However, by means of (the product version of) Ramsey's theorem we can always pass to subsets of K, L, and M for which one of the properties holds (see, e.g., the proof of Corollary 4.7 below).

The next lemma shows that in reduced hypergraphs that do not support $K_5^{(3)}$ at most one pair from a triple of inhabited transversals can have an intersecting link.

Lemma 4.6. *Let $\delta > 0$, let $\mathcal{A}$ be a reduced hypergraph with index set I, and for sufficiently large disjoint sets $K, L, M \subseteq I$ let $\mathcal{Q}(K, L)\mathcal{R}(K, M)\mathcal{S}(L, M)$ be an inhabited triple of transversals. If both pairs of transversals $\mathcal{Q}(K, L)\mathcal{R}(K, M)$ and $\mathcal{Q}(K, L)\mathcal{S}(L, M)$ have δ-intersecting links, then $\mathcal{A}$ supports a $K_5^{(3)}$.*

Proof. Fix $m \in M$, a subset $K_\star \subseteq K$ of size $\lfloor \delta^{-1} \rfloor + 1$, and $q = \binom{\lfloor \delta^{-1} \rfloor + 1}{2}$. Consider an arbitrary pair of distinct indices $k, k' \in K_\star$. Since $|N(Q^{k\ell}, Q^{k'\ell}) \cap N(R^{km}, R^{k'm})| \geqslant \delta|\mathcal{P}^{kk'}|$ for every $\ell \in L$, there is a subset $L_1 \subseteq L$ of size at least $\delta|L|$ such that

$$\bigcap_{\ell \in L_1} N(Q^{k\ell}, Q^{k'\ell}) \cap N(R^{km}, R^{k'm}) \neq \varnothing\,. \tag{4.5}$$

As the pair kk' was taken arbitrarily, we can repeat the argument iteratively q times (once for every pair in $K_\star^{(2)}$) and find nested subsets $L \supseteq L_1 \supseteq L_2 \supseteq \dots \supseteq L_q$ such that (4.5) with L_1 replaced by L_q holds for every $kk' \in K_\star^{(2)}$.

Moreover, we have $|L_q| \geqslant \delta^q|L|$ and, since L is sufficiently large, this yields $|L_q| \geqslant 2$ and we can select $\ell\ell' \in L_q^{(2)}$. Owing to (4.5) with L_1 replaced by L_q, for every $kk' \in K_\star^{(2)}$ there is a vertex $P^{kk'} \in \mathcal{P}^{kk'}$ such that

$$P^{kk'}Q^{k\ell}Q^{k'\ell},\ P^{kk'}Q^{k\ell'}Q^{k'\ell'},\ P^{kk'}R^{km}R^{k'm} \in E(\mathcal{A})\,. \tag{4.6}$$

Next, since $\mathcal{Q}(K, L)\mathcal{S}(L, M)$ has δ-intersecting links and $|K_\star| > \delta^{-1}$, there exists a pair $kk' \in K_\star^{(2)}$ such that

$$N(Q^{k\ell}, Q^{k\ell'}) \cap N(Q^{k'\ell}, Q^{k'\ell'}) \cap N(S^{\ell m}, S^{\ell' m}) \neq \varnothing\,.$$

Therefore, there is a vertex $P^{\ell\ell'} \in \mathcal{P}^{\ell\ell'}$ such that

$$P^{\ell\ell'}Q^{k\ell}Q^{k\ell'},\ P^{\ell\ell'}Q^{k'\ell}Q^{k'\ell'},\ P^{\ell\ell'}S^{\ell m}S^{\ell' m} \in E(\mathcal{A})\,. \tag{4.7}$$

Finally, since $\mathcal{Q}(K,L)\mathcal{R}(K,M)\mathcal{S}(L,M)$ is inhabited, we have

$$Q^{k\ell}R^{km}S^{\ell m},\ Q^{k\ell'}R^{km}S^{\ell' m},\ Q^{k'\ell}R^{k'm}S^{\ell m},\ Q^{k'\ell'}R^{k'm}S^{\ell' m} \in E(\mathcal{A})\,. \tag{4.8}$$

Altogether the ten hyperedges provided by (4.6) – (4.8) show that the vertices $P^{kk'}$, $P^{\ell\ell'}$, together with $Q^{k\ell}$, $Q^{k\ell'}$, $Q^{k'\ell}$, $Q^{k'\ell'}$, R^{km}, $R^{k'm}$, and $S^{\ell m}$, $S^{\ell' m}$ support a $K_5^{(3)}$ on the five indices k, k', ℓ, ℓ', and m. □

By means of the product Ramsey theorem (see, e.g., [7, Theorem 5.1.5]) we can move from at most one pair with intersecting links (given by Lemma 4.6) to at least two pairs with essentially disjoint links.

Corollary 4.7. *Let $t \in \mathbb{N}$, $\delta > 0$, let $\mathcal{A}$ be a reduced hypergraph with index set I that does not support $K_5^{(3)}$, and let $\mathcal{Q}(K,L)\mathcal{R}(K,M)\mathcal{S}(L,M)$ be an inhabited triple of transversals for sufficiently large disjoint sets K, L, $M \subseteq I$.*

Then there exist subsets $K_\star \subseteq K$, $L_\star \subseteq L$, and $M_\star \subseteq M$ each of size t such that at most one pair of restricted transversals $\mathcal{Q}(K_\star, L_\star)\mathcal{R}(K_\star, M_\star)$, $\mathcal{Q}(K_\star, L_\star)\mathcal{S}(L_\star, M_\star)$, $\mathcal{R}(K_\star, M_\star)\mathcal{S}(L_\star, M_\star)$ has δ-intersecting links and all other pairs have δ-disjoint links.

Proof. Define a 2-colouring on the triples $(kk', \ell, m) \in K^{(2)} \times L \times M$ depending on whether $N(Q^{k\ell}, Q^{k'\ell}) \cap N(R^{km}, R^{k'm}) \geqslant \delta|\mathcal{P}^{kk'}|$ or not. Since K, L, and M are large enough, we can deduce from the product Ramsey theorem that there exist large subsets $K_1 \subseteq K$, $L_1 \subseteq L$, and $M_1 \subseteq M$ for which the pair of restricted transversals $\mathcal{Q}(K_1, L_1)\mathcal{R}(K_1, M_1)$ has δ-intersecting or δ-disjoint links.

We can repeat this argument and consider the triples in $L_1^{(2)} \times K_1 \times M_1$ to obtain subsets $K_2 \subseteq K_1, L_2 \subseteq L_1$, and $M_2 \subseteq M_1$ such that the pair $\mathcal{Q}(K_2, L_2)\mathcal{S}(L_2, M_2)$ has δ-intersecting or δ-disjoint links. Observe that these properties are preserved under taking subsets of indices and, hence, also the pair $\mathcal{Q}(K_2, L_2)\mathcal{R}(K_2, M_2)$ has δ-intersecting or δ-disjoint links.

Repeating the Ramsey argument again yields subsets $K_\star \subseteq K_2, L_\star \subseteq L_2$, and $M_\star \subseteq M_2$ such that all pairs of restricted transversals $\mathcal{Q}(K_\star, L_\star)$, $\mathcal{R}(K_\star, M_\star)$, and $\mathcal{S}(L_\star, M_\star)$ have δ-intersecting or δ-disjoint links. Since the initial sets K, L, and M are large enough, we argue that $K_\star$, $L_\star$, and $M_\star$ can be taken of size at least t.

Finally, applying Lemma 4.6 we observe that at most one of those pairs of transversals has a δ-intersecting link, and hence, at least two of them have δ-disjoint links. □

Finally, we may combine Corollaries 4.4 and 4.7. More precisely, after an application of Corollary 4.7 and three consecutive applications of Corollary 4.4 we arrive at the following statement.

Corollary 4.8. *Let $t \in \mathbb{N}$, δ, μ, $d > 0$, let $\mathcal{A}$ be a $(d, \boldsymbol{\wedge})$-dense reduced hypergraph with index set I that does not support a $K_5^{(3)}$, and for sufficiently large disjoint sets K, L, $M \subseteq I$ let $\mathcal{Q}(K,L)\mathcal{R}(K,M)\mathcal{S}(L,M)$ be an inhabited triple of transversals.*

There exist subsets $K_\star \subseteq K$, $L_\star \subseteq L$, and $M_\star \subseteq M$ of size at least t such that

- (i) *at most one pair $\mathcal{Q}(K_\star, L_\star)\mathcal{R}(K_\star, M_\star)$, $\mathcal{Q}(K_\star, L_\star)\mathcal{S}(L_\star, M_\star)$, $\mathcal{R}(K_\star, M_\star)\mathcal{S}(L_\star, M_\star)$ of restricted transversals has δ-intersecting links and all other pairs have δ-disjoint links*
- (ii) *and for every $k \in K_\star$, $\ell \in L_\star$, and $m \in M_\star$ the links $\Lambda(\mathcal{Q}, K_\star, \ell)$, $\Lambda(\mathcal{Q}, L_\star, k)$, $\Lambda(\mathcal{R}, K_\star, m)$, $\Lambda(\mathcal{R}, M_\star, k)$, $\Lambda(\mathcal{S}, L_\star, m)$, and $\Lambda(\mathcal{S}, M_\star, \ell)$ are (μ, d)-holes.* □

4.4. **Equivalent holes.** Roughly speaking, in the next step of the proof of Proposition 2.6 we show that for wicked reduced hypergraphs (see Definition 2.5), the set of holes with width bigger than 1/3 splits into only two classes defined by δ-intersections. For that we generalise the notion of being δ-intersecting from links to holes.

Definition 4.9. *Given a reduced hypergraph $\mathcal{A}$ with index set I, a subset $J \subseteq I$, and μ, $\delta > 0$, we say two μ-holes Φ and Ψ on J are δ-*intersecting *if*

$$\left|\Phi^{ij} \cap \Psi^{ij}\right| \geqslant \delta\left|\mathcal{P}^{ij}\right| \tag{4.9}$$

*for all $ij \in J^{(2)}$. If, on the other hand, (4.9) fails for all $ij \in J^{(2)}$, then we say Φ and Ψ are δ-*disjoint.

For $\mu > 0$ and $\delta \in (0,1]$ the notion of being δ-intersecting defines a reflexive and symmetric relation on the μ-holes on J. Perhaps somewhat surprisingly, the next lemma shows that this relation is also transitive on holes with width bigger than 1/3 in wicked reduced hypergraphs, if one passes to an appropriate subset of J. This justifies the shorthand notation

$$\Phi \equiv_{\delta,J} \Psi$$

for δ-intersecting holes on J. Similarly, $\Phi \not\equiv_{\delta,J} \Psi$ will indicate that Φ and Ψ are δ-disjoint on J. Notice that this statement is stronger than the mere negation of $\Phi \equiv_{\delta,J} \Psi$.

Lemma 4.10 (transitivity lemma)**.** *For every $\varepsilon > 0$ there exists $\mu > 0$ such that for every $t \in \mathbb{N}$ the following holds. Suppose $\mathcal{A}$ is an ε-wicked reduced hypergraph with index set I and for sufficiently large $J \subseteq I$ we are given $(\mu, 1/3 + \varepsilon)$-holes Φ, Ψ, and Ω on J. If*

$$\Phi \equiv_{\varepsilon,J} \Psi \qquad \text{and} \qquad \Psi \equiv_{\varepsilon,J} \Omega\,,$$

then there is a subset $J_\star \subseteq J$ of size at least t such that $\Phi \equiv_{\varepsilon,J_\star} \Omega$.

Proof. Given $\varepsilon > 0$ we fix auxiliary integers t_1, t_2, t_3, and $\mu > 0$ satisfying the hierarchy

$$\varepsilon^{-1} \ll t_3 \ll t_2 \ll t_1,\ \mu^{-1}\,.$$

Let $t \in \mathbb{N}$ and let $\mathcal{A}$ be an ε-wicked reduced hypergraph with index set I and for sufficiently large $J \subseteq I$ let Φ, Ψ, and Ω be $(\mu, 1/3+\varepsilon)$-holes on J such that the pairs $\Phi\Psi$ and $\Psi\Omega$ are ε-intersecting.

Consider an auxiliary 2-colouring of the pairs $ij \in J^{(2)}$ depending on whether

$$|\Phi^{ij} \cap \Omega^{ij}| < \varepsilon|\mathcal{P}^{ij}| \tag{4.10}$$

or not. Since $|J| \longrightarrow (t_1, t)^2_2$, there either exists the desired set $J_\star$, or there is a subset $J_1 \subseteq J$ of size t_1 such that (4.10) holds for every $ij \in J_1^{(2)}$. So it suffices to show that the second possibility contradicts the wickedness of $\mathcal{A}$.

First we note that for all $i < j < k$ from J_1 and every $P^{ij} \in \mathcal{P}^{ij}$ and $P^{jk} \in \mathcal{P}^{jk}$ the $(1/3+\varepsilon, \boldsymbol{\wedge})$-density of $\mathcal{A}$ and the given width of the holes Φ and Ω together with (4.10) imply

$$\begin{aligned}\big|N(P^{ij}, P^{jk}) \cap (\Phi^{ik} \cup \Omega^{ik})\big| &\geqslant \big|N(P^{ij}, P^{jk})\big| + \big|\Phi^{ik}\big| + \big|\Omega^{ik}\big| - \big|\mathcal{P}^{ik}\big| - \big|\Phi^{ik} \cap \Omega^{ik}\big| \\ &\geqslant 2\varepsilon\big|\mathcal{P}^{ik}\big|. \end{aligned} \tag{4.11}$$

We define the reduced subhypergraph $\mathcal{A}_1 \subseteq \mathcal{A}$ with index set J_1, vertex classes $\mathcal{P}^{ij}$ inherited from $\mathcal{A}$, and constituents

$$\mathcal{A}_1^{ijk} = \mathcal{A}^{ijk}[\Phi^{ij} \cap \Psi^{ij}, \Phi^{ik} \cup \Omega^{ik}, \Psi^{jk} \cap \Omega^{jk}]\,.$$

Since the pairs $\Phi\Psi$ and $\Psi\Omega$ are ε-intersecting, we infer from (4.11) for all $i < j < k$ in J_1 that

$$e(\mathcal{A}_1^{ijk}) = \sum_{\substack{P^{ij} \in \Phi^{ij} \cap \Psi^{ij} \\ P^{jk} \in \Psi^{jk} \cap \Omega^{jk}}} \big|N_{\mathcal{A}}(P^{ij}, P^{jk}) \cap (\Phi^{ik} \cup \Omega^{ik})\big| \geqslant 2\varepsilon^3|\mathcal{P}^{ij}||\mathcal{P}^{ik}||\mathcal{P}^{jk}|$$

and, hence, $\mathcal{A}_1$ is $(2\varepsilon^3, \therefore)$-dense.

We consider the ε-exceptional left and right cherries (see Definition 4.2) of the holes Φ, Ψ, and Ω (restricted to J_1) and for every $i < j < k$ in J_1 we set

$$\mathscr{L}^{ijk} = \mathscr{L}^{ijk}(\Psi, \varepsilon) \cup \mathscr{L}^{ijk}(\Omega, \varepsilon) \quad \text{and} \quad \mathscr{R}^{ijk} = \mathscr{R}^{ijk}(\Phi, \varepsilon) \cup \mathscr{R}^{ijk}(\Psi, \varepsilon)\,.$$

We infer from (4.1) that

$$|\mathscr{L}^{ijk}| \leqslant \frac{2\mu}{\varepsilon}|\mathcal{P}^{ij}||\mathcal{P}^{ik}| \quad \text{and} \quad |\mathscr{R}^{ijk}| \leqslant \frac{2\mu}{\varepsilon}|\mathcal{P}^{ik}||\mathcal{P}^{jk}|\,.$$

By the choice of μ we can apply Lemma 3.4 to $\mathcal{A}_1$ with t_2, $2\varepsilon^3$, and $\frac{2\mu}{\varepsilon}$ in place of t, μ and μ'. This yields a set $J_2 \subseteq J_1$ of size t_2 and an inhabited triple of transversals $\mathcal{Q}(J_2)\mathcal{R}(J_2)\mathcal{S}(J_2)$ avoiding the exceptional cherries from $\mathscr{L}^{ijk}$ and $\mathscr{R}^{ijk}$ for every $ijk \in J_2^{(3)}$. In particular, for all $i < j < k$ in J_2 we have

$$Q^{ij}R^{ik}S^{jk} \in E(\mathcal{A}_1^{ijk}) = E\big(\mathcal{A}^{ijk}[\Phi^{ij} \cap \Psi^{ij}, \Phi^{ik} \cup \Omega^{ik}, \Psi^{jk} \cap \Omega^{jk}]\big)\,. \tag{4.12}$$

We fix disjoint subsets $K' < L < M'$ of J_2, where K' and M' have size $\lfloor t_2/3 \rfloor$ and L has size t_3.

Note that by definition $\mathcal{R}(K', M') \subseteq \Phi \cup \Omega$. Due to the product Ramsey theorem, applied with the set of colours $\{\Phi, \Omega\}$, this leads to sets $K \subseteq K'$ and $M \subseteq M'$ of size t_3 and to a hole $\Pi \in \{\Phi, \Omega\}$ such that

$$R^{km} \in \Pi^{km} \text{ for every } k \in K, \text{ and } m \in M. \tag{4.13}$$

Owing to (4.12) the restricted transversals $\mathcal{Q}(K, L)$, $\mathcal{R}(K, M)$, and $\mathcal{S}(L, M)$ form an inhabited triple in $\mathcal{A}$. We derive a contradiction by Lemma 4.6 and for that we shall show that two of the pairs $\mathcal{Q}(K, L)\mathcal{R}(K, M)$, $\mathcal{Q}(K, L)\mathcal{S}(L, M)$, and $\mathcal{R}(K, M)\mathcal{S}(L, M)$ have ε-intersecting links.

First, we recall that, independent of the chosen Π, the pair $\mathcal{Q}(K, L)\mathcal{S}(L, M)$ consists of transversals inside the hole Ψ and both avoid the exceptional left and right cherries from Ψ. Hence, for all $k \in K$, $\ell\ell' \in L^{(2)}$, and $m \in M$ we have

$$\big|N_{\mathcal{A}}(Q^{k\ell}, Q^{k\ell'}) \cap \Psi^{\ell\ell''}\big| < \varepsilon\big|\mathcal{P}^{\ell\ell'}\big| \quad \text{and} \quad \big|N_{\mathcal{A}}(S^{\ell m}, S^{\ell' m}) \cap \Psi^{\ell\ell'}\big| < \varepsilon\big|\mathcal{P}^{\ell\ell'}\big|.$$

Consequently, the $(1/3 + \varepsilon, \boldsymbol{\wedge})$-density of $\mathcal{A}$ and the width of Ψ imply

$$\big|N_{\mathcal{A}}(Q^{k\ell}, Q^{k\ell'}) \cap N_{\mathcal{A}}(S^{\ell m}, S^{\ell' m})\big| > \varepsilon\big|\mathcal{P}^{\ell\ell'}\big|$$

for every $k \in K$, $\ell\ell' \in L^{(2)}$, and $m \in M$, i.e., the pair $\mathcal{Q}(K, L)\mathcal{S}(L, M)$ has ε-intersecting links.

If $\Pi = \Phi$, then $\mathcal{Q}(K, L)$ and $\mathcal{R}(K, M)$ are both transversals in Φ (see (4.13)) and both $\mathcal{Q}$ and $\mathcal{R}$ avoid the exceptional right cherries of Φ. As before, this implies that the pair $\mathcal{Q}(K, L)\mathcal{R}(K, M)$ has ε-intersecting links. So Lemma 4.6 tells us that $\mathcal{A}$ supports a $K_5^{(3)}$, contrary to the wickedness of $\mathcal{A}$.

Analogously, if $\Pi = \Omega$, then $\mathcal{R}(K, M)$ and $\mathcal{S}(L, M)$ are both transversals in Ω and, since both $\mathcal{R}$ and $\mathcal{S}$ avoid the exceptional left cherries of Ω, the pair of transversals has ε-intersecting links, which leads to the same contradiction. □

Another application of Ramsey's theorem leads to the following corollary.

Corollary 4.11. *For every $\varepsilon \in (0, 1]$ there exists $\mu > 0$ such that for all integers t, $r \geqslant 2$ the following holds. Suppose $\mathcal{A}$ is an ε-wicked reduced hypergraph with index set I and for sufficiently large $J \subseteq I$ we are given $(\mu, 1/3 + \varepsilon)$-holes $\Phi_1, \dots, \Phi_r$ on J.*

Then there is a subset $J_\star \subseteq J$ of size t such that

- *(i) for all ϱ, $\varrho' \in [r]$ the holes Φ_ϱ and $\Phi_{\varrho'}$ are either ε-intersecting or ε-disjoint on $J_\star$*
- *(ii) and $\equiv_{\varepsilon, J_\star}$ is an equivalence relation on $\{\Phi_1, \dots, \Phi_r\}$ with at most two equivalence classes.*

Proof. For $\varepsilon \in (0, 1]$ let $\mu > 0$ be given by Lemma 4.10. For fixed t, $r \geqslant 2$ let $t' \geqslant t$ be sufficiently large for an application of Lemma 4.10 with ε, μ, and with 2 in place of t.

For a given ε-wicked reduced hypergraph $\mathcal{A}$ and $(\mu, 1/3 + \varepsilon)$-holes $\Phi_1, \dots, \Phi_r$ we impose that the size of J is at least the $2^{\binom{r}{2}}$-colour Ramsey number for graph cliques on t' vertices, i.e.,

$$|J| \longrightarrow (t')^2_{|\Xi|} \quad \text{for} \quad \Xi = \big\{\xi = (\xi_{\varrho\varrho'})_{\varrho\varrho' \in [r]^{(2)}} \colon \xi_{\varrho\varrho'} \in \{0,1\} \text{ for } \varrho\varrho' \in [r]^{(2)}\big\}\,. \tag{4.14}$$

We assign to a pair $ij \in J^{(2)}$ the colour $\xi = (\xi_{\varrho\varrho'})_{\varrho\varrho' \in [r]^{(2)}}$ with $\xi_{\varrho\varrho'} = 1$ signifying

$$\big|\Phi^{ij}_{\varrho} \cap \Phi^{ij}_{\varrho'}\big| \geqslant \varepsilon\big|\mathcal{P}^{ij}\big|$$

and $\xi_{\varrho\varrho'} = 0$ otherwise. Owing to (4.14) there exists a subset $J_\star \subseteq J$ of size at least $t' \geqslant t$ and a colour $\xi^\star = (\xi^\star_{\varrho\varrho'})_{\varrho\varrho' \in [r]^{(2)}}$ such that all pairs of $J_\star$ were assigned $\xi^\star$. Note that assertion *(i)* follows directly from the definition of the colouring, i.e., Φ_ϱ and $\Phi_{\varrho'}$ are ε-intersecting on $J_\star$ if $\xi^\star_{\varrho\varrho'} = 1$ and ε-disjoint otherwise.

Obviously the relation $\equiv_{\varepsilon, J_\star}$ is reflexive and symmetric. Moreover, our choice of t' allows us to invoke Lemma 4.10 and the transitivity follows from the definition of the colouring. Since all holes have width at least $1/3 + \varepsilon$, at least two among any choice of three holes must share at least $\varepsilon|\mathcal{P}^{ij}|$ vertices in $\mathcal{P}^{ij}$ for any $ij \in J_\star^{(2)}$ and, hence, $\equiv_{\varepsilon, J_\star}$ has at most two equivalence classes. $\square$

It will later be important that, under sufficiently general circumstances, there really are two distinct equivalence classes.

Lemma 4.12. *Given $t \in \mathbb{N}$ and $\varepsilon, \mu > 0$ let I be a sufficiently large set of indices. For every ε-wicked reduced hypergraph $\mathcal{A}$ with index set I there are a set $J \subseteq I$ of size t and two ε-disjoint $(\mu, 1/3 + \varepsilon)$-holes on J.*

Proof. We may assume that we have an integer t' fitting into the hierarchy

$$|I| \gg t' \gg t, \mu^{-1}\,.$$

Since $\mathcal{A}$ is, in particular, $(1/3 + \varepsilon, \therefore)$-dense, Theorem 3.2 applied with $3t'$, $1/3 + \varepsilon$ here in place of t, μ there yields a set $I' \subseteq I$ of size $3t'$ and an inhabited triple of transversals $\mathcal{Q}(I')\mathcal{R}(I')\mathcal{S}(I')$.

Fix an arbitrary partition $I' = K' \cupdot L' \cupdot M'$ such that $|K'| = |L'| = |M'| = t'$. Now we apply Corollary 4.8 with ε, $1/3 + \varepsilon$ here in place of δ, d there to the inhabited triple of restricted transversals $\mathcal{Q}(K', L')\mathcal{R}(K', M')\mathcal{S}(L', M')$. This yields subsets $K \subseteq K'$, $L \subseteq L'$, and $M \subseteq M'$ of size t satisfying properties *(i)* and *(ii)* of the corollary.

By *(i)* we may assume without loss of generality that the pair $\mathcal{Q}(K, L)\mathcal{R}(K, M)$ has ε-disjoint links. Thus, fixing $\ell \in L$ and $m \in M$ arbitrarily we obtain the desired ε-disjoint $(\mu, 1/3 + \varepsilon)$-holes $\Lambda(\mathcal{Q}, K, \ell)$ and $\Lambda(\mathcal{R}, K, m)$ on $J = K$. $\square$

4.5. **Unions of equivalent holes.** We proceed with the union lemma, which roughly speaking asserts that unions of equivalent holes are holes. As usual, the precise statement involves a considerable loss of relevant indices. Moreover, if want such a union $\Phi \cup \Psi$ to be a μ-hole, we need to assume that Φ and Ψ themselves are ν-holes for some very small $\nu \ll \mu$.

Lemma 4.13 (union lemma)**.** *For every μ, $\varepsilon > 0$ there exists $\nu > 0$ such that for every $t \in \mathbb{N}$ the following holds. Suppose $\mathcal{A}$ is an ε-wicked reduced hypergraph with index set I and for a sufficiently large subset $J \subseteq I$ we are given two $(\nu, 1/3+\varepsilon)$-holes Φ and Ψ on J such that $\Phi \equiv_{\varepsilon,J} \Psi$.*

Then, there exists a subset $J_\star \subseteq J$ of size at least t such that $\Phi \cup \Psi$ is a μ-hole on $J_\star$.

Proof. As decreasing μ makes the lemma stronger, we may assume that $\mu \ll \varepsilon$. Now we take auxiliary integers t_1, t_2, t_3, and t_4 and a positive real ν fitting into the hierarchy

$$\varepsilon \gg t_4^{-1} \gg t_3^{-1} \gg t_2^{-1}, \nu \gg t_1^{-1}\,.$$

More precisely we assume that

(*1*) t_4 is so large that the conclusion of Corollary 4.11 holds for ε, μ, and for 2, 4, t_4 here in place of t, r, $|J|$ there;

(*2*) t_3 is so large that the conclusion of Corollary 4.8 holds for t_4, ε, μ, $1/3+\varepsilon$, t_3 here in place of t, δ, μ, d, $\min\{|K|, |L|, |M|\}$ there;

(*3*) t_2 is so large and $\nu \leqslant \mu$ is so small that the conclusion of Lemma 3.4 holds for $3t_3$, $\mu/8$, $2\nu/\varepsilon$, t_2 here in place of t, μ, μ', $|J|$ there;

(*4*) and $t_1 \longrightarrow (t_2)^3_8$.

Finally, given $t \in \mathbb{N}$ we suppose that $J \subseteq I$ is large so that

$$|J| \longrightarrow (t_1, t)^3_2\,.$$

For $(\nu, 1/3+\varepsilon)$-holes Φ and Ψ on J let

$$\mathscr{L} = \mathscr{L}(\Phi,\varepsilon) \cup \mathscr{L}(\Psi,\varepsilon) \quad \text{and} \quad \mathscr{R} = \mathscr{R}(\Phi,\varepsilon) \cup \mathscr{R}(\Psi,\varepsilon)$$

be their ε-exceptional left and right cherries. For later reference we recall that (4.1) yields

$$|\mathscr{L}^{ijk}| \leqslant \frac{2\nu}{\varepsilon}|\mathcal{P}^{ij}||\mathcal{P}^{ik}| \quad \text{and} \quad |\mathscr{R}^{ijk}| \leqslant \frac{2\nu}{\varepsilon}|\mathcal{P}^{ik}||\mathcal{P}^{jk}|\,. \tag{4.15}$$

We begin with an application of Ramsey's theorem for hypergraphs and consider a 2-colouring of the triples $ijk \in J^{(3)}$ depending on whether

$$e(\Phi^{ij} \cup \Psi^{ij}, \Phi^{ik} \cup \Psi^{ik}, \Phi^{jk} \cup \Psi^{jk}) > \mu|\mathcal{P}^{ij}||\mathcal{P}^{ik}||\mathcal{P}^{jk}| \tag{4.16}$$

or not. Owing to the size of J, there either exists the desired set $J_\star$, or there is a subset $J_1 \subseteq J$ of size t_1 such that (4.16) holds for all $ijk \in J_1^{(3)}$. We shall show that the second case leads to a contradiction.

First we observe that for every $ijk \in J_1^{(3)}$ inequality (4.16) implies that for at least one of the eight possible triples $(\Pi_1, \Pi_2, \Pi_3) \in \{\Phi, \Psi\}^3$ we have

$$e(\Pi_1^{ij}, \Pi_2^{ik}, \Pi_3^{jk}) > \frac{\mu}{8}|\mathcal{P}^{ij}||\mathcal{P}^{ik}||\mathcal{P}^{jk}|\,. \tag{4.17}$$

(Actually, since Φ and Ψ are ν-holes and $\nu \leqslant \mu/8$, inequality (4.17) can neither hold for $e(\Phi^{ij}, \Phi^{ik}, \Phi^{jk})$ nor for $e(\Psi^{ij}, \Psi^{ik}, \Psi^{jk})$, but we shall not use this here.) Thus, there exists an 8-colouring of $J_1^{(3)}$ such that if the colour of a triple $ijk \in J_1^{(3)}$ is $(\Pi_1, \Pi_2, \Pi_3) \in \{\Phi, \Psi\}^3$, then this indicates the validity of (4.17) for this triple of holes. In view of *(4)* there are a subset $J_2 \subseteq J_1$ of size t_2 and a fixed colour $(\Pi_1, \Pi_2, \Pi_3) \in \{\Phi, \Psi\}^3$ such that inequality (4.17) holds for every $ijk \in J_2^{(3)}$.

Now the reduced subhypergraph $\mathcal{A}_2 \subseteq \mathcal{A}$ with index set J_2, vertex classes inherited from $\mathcal{A}$, and constituents

$$\mathcal{A}_2^{ijk} = \mathcal{A}^{ijk}[\Pi_1^{ij}, \Pi_2^{ik}, \Pi_3^{jk}] \tag{4.18}$$

for all $i < j < k$ in J_2 is $(\mu/8, \therefore)$-dense. Owing to (4.15) and our choice of t_2 and ν in *(3)*, Lemma 3.4 ensures that there are a subset $J_3 \subseteq J_2$ of size $3t_3$ and an inhabited triple of transversals $\mathcal{Q}(J_3)\mathcal{R}(J_3)\mathcal{S}(J_3)$ where each transversal avoids the sets of exceptional left and right cherries $\mathscr{L}$ and $\mathscr{R}$ of Φ and Ψ.

Since $\mathcal{Q}(J_3)\mathcal{R}(J_3)\mathcal{S}(J_3)$ is an inhabited triple, we have $Q^{ij}R^{ik}S^{jk} \in E(\mathcal{A}_2)$ for every $i < j < k$ in J_3 and, therefore, (4.18) implies

$$Q^{ij} \in \Pi_1\,, \quad R^{ik} \in \Pi_2\,, \quad \text{and} \quad S^{jk} \in \Pi_3 \tag{4.19}$$

for all $i < j < k$ in J_3.

Fix disjoint subsets of indices $K_3 < L_3 < M_3$ of J_3 each of size t_3. Clearly, the triple of restricted transversals $\mathcal{Q}(K_3, L_3)\mathcal{R}(K_3, M_3)\mathcal{S}(L_3, M_3)$ is still inhabited. Therefore, the choice of t_3 in *(2)* allows an application of Corollary 4.8, which yields subsets $K_4 \subseteq K_3$, $L_4 \subseteq L_3$, and $M_4 \subseteq M_3$ each of size t_4 satisfying properties *(i)* and *(ii)* of Corollary 4.8.

Next we shall show that all three pairs of restricted transversals $\mathcal{Q}(K_4, L_4)\mathcal{R}(K_4, M_4)$, $\mathcal{Q}(K_4, L_4)\mathcal{S}(L_4, M_4)$, and $\mathcal{R}(K_4, M_4)\mathcal{S}(L_4, M_4)$ have ε-intersecting links. However, this contradicts property *(i)* of Corollary 4.8, which allows only one pair of transversals with ε-intersecting links and this contradiction concludes the proof of Lemma 4.13. Below we show that the pair $\mathcal{Q}(K_4, L_4)\mathcal{R}(K_4, M_4)$ has ε-intersecting links. The proof for the other pairs follows verbatim the same lines.

Fix some $\ell \in L_4$ and $m \in M_4$. Property *(ii)* of Corollary 4.8 tells us that $\Lambda(\mathcal{Q}, K_4, \ell)$ and $\Lambda(\mathcal{R}, K_4, m)$ are $(\mu, 1/3 + \varepsilon)$-holes on K_4. Moreover, since $\nu \leqslant \mu$, also Φ and Ψ are $(\mu, 1/3 + \varepsilon)$-holes on K_4 and, therefore, the choice of t_4 in *(1)* and an application of Corollary 4.11 yield a subset $K_\star \subseteq K_4$ of size two such that $\equiv = \equiv_{\varepsilon, K_\star}$ is an equivalence relation with at most two equivalence classes on the μ-holes

$$\Lambda(\mathcal{Q}, K_\star, \ell)\,, \quad \Lambda(\mathcal{R}, K_\star, m)\,, \quad \Pi_1\,, \quad \text{and} \quad \Pi_2\,.$$

In view of (4.19) we have $\mathcal{Q}(K_\star, L_4) \subseteq \Pi_1$ and $\mathcal{R}(K_\star, M_4) \subseteq \Pi_2$ and, since $\mathcal{Q}$ and $\mathcal{R}$ avoid the exceptional cherries from $\mathscr{L}$ and $\mathscr{R}$, we infer

$$|N(Q^{k\ell}, Q^{k'\ell}) \cap \Pi_1^{kk'}| < \varepsilon|\mathcal{P}^{kk'}| \qquad \text{and} \qquad |N(R^{km}, R^{k'm}) \cap \Pi_2^{kk'}| < \varepsilon|\mathcal{P}^{kk'}|\,,$$

where k and k' denote the two elements of $K_\star$. Consequently,

$$\Pi_1 \not\equiv \Lambda(\mathcal{Q}, K_\star, \ell) \quad \text{and} \quad \Pi_2 \not\equiv \Lambda(\mathcal{R}, K_\star, m)\,.$$

As our assumption $\Phi \equiv \Psi$ yields $\Pi_1 \equiv \Pi_2$ and $\equiv$ has at most two equivalence classes, we thus arrive at

$$\Lambda(\mathcal{Q}, K_\star, \ell) \equiv \Lambda(\mathcal{R}, K_\star, m)\,.$$

In other words we have $|N(Q^{k\ell}, Q^{k'\ell}) \cap N(R^{km}, R^{k'm})| \geqslant \varepsilon|\mathcal{P}^{kk'}|$, which excludes the possibility that the pair of transvervals $\mathcal{Q}(K_4, L_4)\mathcal{R}(K_4, M_4)$ has ε-disjoint links. So by property *(i)* of Corollary 4.8 it follows that this pair has ε-intersecting links, as desired. □

For later reference we now state a corollary that follows from Corollary 4.11 and Lemma 4.13.

Corollary 4.14. *For every μ, $\varepsilon > 0$ there exists $\nu > 0$ such that for every $t \in \mathbb{N}$ the following holds. Suppose $\mathcal{A}$ is an ε-wicked reduced hypergraph with index set I and for a sufficiently large subset $J \subseteq I$ we are given three $(\nu, 1/3 + \varepsilon)$-holes Φ, Ψ, and Ω on J such that Φ and Ψ are ε-disjoint.*

Then, there exists a subset $J_\star \subseteq J$ of size at least t such that

(A) either $\Phi \cup \Omega$ is a $(\mu, 1/3 + \varepsilon)$-hole ε-disjoint with Ψ
(B) or $\Psi \cup \Omega$ is a $(\mu, 1/3 + \varepsilon)$-hole ε-disjoint with Φ.

Proof. Again we may assume that $\mu \ll \varepsilon$. Take appropriate constants

$$\nu \ll \mu \quad \text{and} \quad t_1 \gg t_2 \gg t, \nu^{-1}\,,$$

and assume that $|J| \gg t_1$.

Due to Corollary 4.11 there is a subset $J_1 \subseteq J$ of size t_1 such that $\equiv_{\varepsilon, J_1}$ is an equivalence relation with at most two equivalence classes on $\{\Phi, \Psi, \Omega\}$. By hypothesis the holes Φ and Ψ are in different classes and thus we may assume without loss of generality that

$$\Omega \equiv_{\varepsilon, J_1} \Phi \qquad \text{and} \qquad \Omega \not\equiv_{\varepsilon, J_1} \Psi\,.$$

An application of Lemma 4.13 yields the existence of a subset $J_2 \subseteq J_1$ of size t_2 on which

$$\Phi \cup \Omega \text{ is a } (\mu, 1/3 + \varepsilon)\text{-hole.}$$

Now a second application of Corollary 4.11 leads to a t-element subset $J_\star \subseteq J_2$ such that $\equiv_{\varepsilon, J_\star}$ is an equivalence relation with at most two equivalence classes on $\{\Phi, \Phi \cup \Omega, \Psi\}$. Since $\Phi \cup \Omega \equiv_{\varepsilon, J_\star} \Phi \not\equiv_{\varepsilon, J_\star} \Psi$, we have $\Phi \cup \Omega \not\equiv_{\varepsilon, J_\star} \Psi$. Altogether both parts of *(A)* hold. □

4.6. **Holes derived from two transversals.** Before we can make further progress, we need to analyse holes generated by two transversals. Given two transversals $\mathcal{Q}(J)$ and $\mathcal{R}(J)$ in a wicked reduced hypergraph $\mathcal{A}$, we wonder whether for fixed $i \in J$ the sets $\Omega_i^{jk} \subseteq \mathcal{P}^{jk}$ defined by $\Omega_i^{jk} = N(Q^{ij}, R^{ik})$ form a hole. There are several possible cases depending on how i, j, k are ordered, and in the lemma that follows we focus on the case $i < j < k$. It turns out that if the links of $\mathcal{Q}$ and $\mathcal{R}$ satisfy a certain equivalence condition (see (4.20) below), then on a large subset of J the sets Ω_i^{jk} form holes.

Lemma 4.15. *Let $\varepsilon > 0$, $\nu > 0$, and $t \in \mathbb{N}$ be given and suppose that $\mathcal{A}$ is an ε-wicked reduced hypergraph with index set I. If $J \subseteq I$ is sufficiently large and $\mathcal{Q}(J)$, $\mathcal{R}(J)$ are two transversals on J such that all $i < j < k < \ell$ from J satisfy*

$$|N(Q^{ik}, Q^{jk}) \cap N(R^{i\ell}, R^{j\ell})| \geqslant \varepsilon|\mathcal{P}^{ij}|\,, \tag{4.20}$$

then there is a set $J_\star \subseteq J$ of size t such that we have

$$e\big(N(Q^{ij}, R^{ik}), N(Q^{ij}, R^{i\ell}), N(Q^{ik}, R^{i\ell})\big) \leqslant \nu|\mathcal{P}^{jk}||\mathcal{P}^{j\ell}||\mathcal{P}^{k\ell}|$$

for all $i < j < k < \ell$ in $J_\star$.

Proof. Suppose that $|J| \gg t_1 \gg t_2 \gg t, \varepsilon^{-1}, \nu^{-1}$. Set $\Omega_i^{jk} = N(Q^{ij}, R^{ik})$ for all $i < j < k$ from J and colour the quadruples $i < j < k < \ell$ depending on whether

$$e(\Omega_i^{jk}, \Omega_i^{j\ell}, \Omega_i^{k\ell}) > \nu|\mathcal{P}^{jk}||\mathcal{P}^{j\ell}||\mathcal{P}^{k\ell}| \tag{4.21}$$

holds or fails. Due to $|J| \longrightarrow (4t_1, t)_2^4$ this either leads to the desired set $J_\star$ of size t, or to a set $J_1 \subseteq J$ of size $4t_1$ such that (4.21) holds for all $i < j < k < \ell$ in J_1.

Let $J_1 = X_1 \cup K_1 \cup L_1 \cup M_1$ be the (unique) partition of J_1 into sets of size t_1 satisfying $X_1 < K_1 < L_1 < M_1$. Now for every $x \in X_1$ the reduced subhypergraph $\mathcal{A}_x$ of $\mathcal{A}$ with index set $K_1 \cup L_1 \cup M_1$, vertex classes inherited from $\mathcal{A}$, and constituents $\mathcal{A}_x^{k\ell m} = \mathcal{A}^{k\ell m}[\Omega_x^{k\ell}, \Omega_x^{km}, \Omega_x^{\ell m}]$ is $(\nu, \therefore)$-tridense. Therefore, Lemma 3.6 applied to t_2, 1, ν here in place of t, r, μ there yields subsets $X_2 \subseteq X_1$, $K_2 \subseteq K_1$, $L_2 \subseteq L_1$, and $M_2 \subseteq M_1$ of size t_2 and a triple of transversals $\mathcal{T}(K_2, L_2)\mathcal{U}(K_2, M_2)\mathcal{V}(L_2, M_2)$ which is inhabited in every $\mathcal{A}_x$ with $x \in X_2$.

Owing to the definition of the constituents of these reduced hypergraphs this means that for all $(x, k, \ell, m) \in X_2 \times K_2 \times L_2 \times M_2$ we have

$$T^{k\ell} \in \Omega_x^{k\ell}\,, \quad U^{km} \in \Omega_x^{km}\,, \quad V^{\ell m} \in \Omega_x^{\ell m}\,, \quad \text{and} \quad T^{k\ell}U^{km}V^{\ell m} \in E(\mathcal{A}^{k\ell m})\,.$$

In other words, all four triples of transversals

$$\mathcal{Q}(X_2, K_2)\mathcal{R}(X_2, L_2)\mathcal{T}(K_2, L_2)\,, \qquad \mathcal{Q}(X_2, K_2)\mathcal{R}(X_2, M_2)\mathcal{U}(K_2, M_2)\,,$$
$$\mathcal{Q}(X_2, L_2)\mathcal{R}(X_2, M_2)\mathcal{V}(L_2, M_2)\,, \qquad \text{and} \qquad \mathcal{T}(K_2, L_2)\mathcal{U}(K_2, M_2)\mathcal{V}(L_2, M_2)$$

are inhabited in $\mathcal{A}$.

We successively apply Corollary 4.8 to these four triples of inhabited transversals with ε, ν, $1/3 + \varepsilon$ here in place of δ, μ, d there. Each of these applications shrinks the sets of indices still under consideration and eventually we obtain sets X_3, K_3, L_3, and M_3 of size 2, which satisfy *(i)* and *(ii)* of Corollary 4.8 for all those four inhabited triples of transversals. Let us write $X_3 = \{x, x'\}$, $K_3 = \{k, k'\}$, $L_3 = \{\ell, \ell'\}$, and $M_3 = \{m, m'\}$.

Now our assumption on the transversals $\mathcal{Q}$ and $\mathcal{R}$ yields

$$|N(Q^{xk}, Q^{x'k}) \cap N(R^{x\ell}, R^{x'\ell})| \geqslant \varepsilon|\mathcal{P}^{xx'}|$$

and thus the pair $\mathcal{Q}(X_3, K_3)\mathcal{R}(X_3, L_3)$ has ε-intersecting links. So by (i) of Corollary 4.8 applied to $\mathcal{QRT}$ the pairs $\mathcal{Q}(X_3, K_3)\mathcal{T}(K_3, L_3)$ and $\mathcal{R}(X_3, L_3)\mathcal{T}(K_3, L_3)$ have ε-disjoint links. Similarly, the pairs $\mathcal{Q}(X_3, K_3)\mathcal{R}(X_3, M_3)$ and $\mathcal{Q}(X_3, L_3)\mathcal{R}(X_3, M_3)$ have ε-intersecting links, whereas the pairs $\mathcal{Q}(X_3, K_3)\mathcal{U}(K_3, M_3)$, $\mathcal{R}(X_3, M_3)\mathcal{U}(K_3, M_3)$, $\mathcal{Q}(X_3, L_3)\mathcal{V}(L_3, M_3)$, and $\mathcal{R}(X_3, M_3)\mathcal{V}(L_3, M_3)$ have ε-disjoint links.

Let us now look at the three subsets $N(Q^{xk}, Q^{xk'})$, $N(T^{k\ell}, T^{k'\ell})$, and $N(U^{km}, U^{k'm})$ of $\mathcal{P}^{kk'}$. As $\mathcal{A}$ is $(1/3 + \varepsilon, \boldsymbol{\wedge})$-dense, each of them has at least the size $(1/3 + \varepsilon)|\mathcal{P}^{kk'}|$. Moreover, the fact that $\mathcal{Q}(X_3, K_3)\mathcal{T}(K_3, L_3)$ and $\mathcal{Q}(X_3, K_3)\mathcal{U}(K_3, M_3)$ have ε-disjoint links implies

$$|N(Q^{xk}, Q^{xk'}) \cap N(T^{k\ell}, T^{k'\ell})| < \varepsilon|\mathcal{P}^{kk'}| \quad \text{and} \quad |N(Q^{xk}, Q^{xk'}) \cap N(U^{km}, U^{k'm})| < \varepsilon|\mathcal{P}^{kk'}|\,.$$

For all these reason we have $|N(T^{k\ell}, T^{k'\ell}) \cap N(U^{km}, U^{k'm})| \geqslant \varepsilon|\mathcal{P}^{kk'}|$ and, hence, the links of $\mathcal{T}(K_3, L_3)\mathcal{U}(K_3, M_3)$ are ε-intersecting.

Arguing similarly with the subsets $N(R^{xm}, R^{xm'})$, $N(U^{km}, U^{km'})$, and $N(V^{\ell m}, V^{\ell m'})$ of $\mathcal{P}^{mm'}$ one can show that the pair $\mathcal{U}(K_3, M_3)\mathcal{V}(L_3, M_3)$ has ε-intersecting links as well. Thus the application of Corollary 4.8 to the triple $\mathcal{TUV}$ yields two pairs of ε-intersecting links, contrary to clause (i). □

We proceed with a related result that, given two transversals $\mathcal{Q}(J)$, $\mathcal{S}(J)$, addresses holes composed of sets of the form $\Omega^{ij}_x = N(Q^{ix}, S^{xj})$, where $i < x < j$. The proof is very similar to the previous one, but towards the end we shall need an additional argument.

Lemma 4.16. *Given $\varepsilon > 0$, $\nu > 0$, and $t \in \mathbb{N}$ let $\mathcal{A}$ be an ε-wicked reduced hypergraph with index set I. If $J \subseteq I$ is sufficiently large and $\mathcal{Q}(J)$, $\mathcal{S}(J)$ are two transversals on J such that all $i < j < k < \ell$ from J satisfy*

$$|N(Q^{ij}, Q^{ik}) \cap N(S^{j\ell}, S^{k\ell})| \geqslant \varepsilon|\mathcal{P}^{jk}| \quad \text{and} \quad |N(S^{ik}, S^{i\ell}) \cap N(S^{jk}, S^{j\ell})| \geqslant \varepsilon|\mathcal{P}^{k\ell}|\,, \tag{4.22}$$

then there is a set $J_\star \subseteq J$ of size t such that we have

$$e\big(N(Q^{ix}, S^{xj}), N(Q^{ix}, S^{xk}), N(Q^{jy}, S^{yk})\big) \leqslant \nu|\mathcal{P}^{ij}||\mathcal{P}^{ik}||\mathcal{P}^{jk}|$$

for all $i < x < j < y < k$ in $J_\star$.

Proof. Since decreasing ν makes the statement stronger, we may assume that $\nu \ll \varepsilon$. Suppose that $|J| \gg t_1 \gg t_2 \gg t_3 \gg t, \varepsilon^{-1}, \nu^{-1}$. This time we set $\Omega^{ij}_x = N(Q^{ix}, S^{xj})$ for all $i < x < j$ from J and colour the quintuples $i < x < j < y < k$ depending on whether

$$e(\Omega^{ij}_x, \Omega^{ik}_x, \Omega^{jk}_y) > \nu|\mathcal{P}^{ij}||\mathcal{P}^{ik}||\mathcal{P}^{jk}| \tag{4.23}$$

holds or fails. Due to $|J| \longrightarrow (5t_1, t)^5_2$ this either leads to the desired set $J_\star$ of size t, or to a set $J_1 \subseteq J$ of size $5t_1$ such that (4.23) holds for all $i < x < j < y < k$ in J_1.

Now we partition

$$J_1 = K_1 \cup X_1 \cup L_1 \cup Y_1 \cup M_1$$

into t_1-sets ordered by $K_1 < X_1 < L_1 < Y_1 < M_1$ and form for every pair $(x, y) \in X_1 \times Y_1$ the reduced subhypergraph $\mathcal{A}_{xy}$ of $\mathcal{A}$ with index set $K_1 \cup L_1 \cup M_1$, vertex classes inherited from $\mathcal{A}$, and with constituents $\mathcal{A}^{k\ell m}_{xy} = \mathcal{A}^{k\ell m}[\Omega^{k\ell}_x, \Omega^{km}_x, \Omega^{\ell m}_y]$. As these reduced hypergraphs are $(\nu, \therefore)$-tridense, Lemma 3.6 applied to t_2, 2, ν here in place of t, r, μ there yields subsets $K_2 \subseteq K_1$, $X_2 \subseteq X_1$, $L_2 \subseteq L_1$, $Y_2 \subseteq Y_1$, and $M_2 \subseteq M_1$ of size t_2 and a triple of transversals $\mathcal{T}(K_2, L_2)\mathcal{U}(K_2, M_2)\mathcal{V}(L_2, M_2)$, which is inhabited in every $\mathcal{A}_{xy}$ with $x \in X_2$ and $y \in Y_2$.

As in the proof of the foregoing lemma one observes that the four triples

$$\mathcal{Q}(K_2, X_2)\mathcal{T}(K_2, L_2)\mathcal{S}(X_2, L_2)\,, \qquad \mathcal{Q}(K_2, X_2)\mathcal{U}(K_2, M_2)\mathcal{S}(X_2, M_2)\,,$$
$$\mathcal{Q}(L_2, Y_2)\mathcal{V}(L_2, M_2)\mathcal{S}(Y_2, M_2) \qquad \text{and} \qquad \mathcal{T}(K_2, L_2)\mathcal{U}(K_2, M_2)\mathcal{V}(L_2, M_2)$$

are inhabited in $\mathcal{A}$.

Again, we apply Corollary 4.8 successively to all these triples, this time obtaining sets K_3, X_3, L_3, Y_3, and M_3 of size t_3, satisfying *(i)* and *(ii)* of Corollary 4.8 for these four triples of transversals. As before the desired contradiction arises from the fact that the pairs $\mathcal{T}(K_3, L_3)\mathcal{U}(K_3, M_3)$ and $\mathcal{U}(K_3, M_3)\mathcal{V}(L_3, M_3)$ have ε-intersecting links, contrary to Corollary 4.8 *(i)*.

The first of these two facts can be proved in the usual way: By (4.22) the pairs $\mathcal{Q}(K_3, X_3)\mathcal{S}(X_3, L_3)$ and $\mathcal{Q}(K_3, X_3)\mathcal{S}(X_3, M_3)$ have ε-intersecting links and, therefore, the pairs $\mathcal{Q}(K_3, X_3)\mathcal{T}(K_3, L_3)$ and $\mathcal{Q}(K_3, X_3)\mathcal{U}(K_3, M_3)$ have ε-disjoint links. So for arbitrary $k, k' \in K_3$, $x \in X_3$, $\ell \in L_3$, and $m \in M_3$ the subsets $N(Q^{kx}, Q^{k'x})$, $N(T^{k\ell}, T^{k'\ell})$, and $N(U^{km}, U^{k'm})$ of $\mathcal{P}^{kk'}$ have at least the size $(1/3 + \varepsilon)|\mathcal{P}^{kk'}|$ and the first of them intersects the two other ones in less than $\varepsilon|\mathcal{P}^{kk'}|$ vertices each. This yields

$$|N(T^{k\ell}, T^{k'\ell}) \cap N(U^{km}, U^{k'm})| > \varepsilon|\mathcal{P}^{kk'}|$$

and thus the pair $\mathcal{T}(K_3, L_3)\mathcal{U}(K_3, M_3)$ has indeed ε-intersecting links.

It is less obvious, however, that the pair $\mathcal{U}(K_3, M_3)\mathcal{V}(L_3, M_3)$ has ε-intersecting links as well. To confirm this, we pick arbitrary vertices $k \in K_3$, $x \in X_3$, $\ell \in L_3$, $y \in Y$. Due to $\nu \ll \varepsilon$ we can invoke Corollary 4.11 and pass to a subset $M_4 \subseteq M_3$ of size 2 such that $\equiv = \equiv_{\varepsilon, M_4}$ is an equivalence relation with at most two equivalence classes on the set of ν-holes

$$\big\{\Lambda(\mathcal{U}, M_4, k), \Lambda(\mathcal{V}, M_4, \ell), \Lambda(\mathcal{S}, M_4, x), \Lambda(\mathcal{S}, M_4, y)\big\}\,.$$

By the left statement in (4.22) the pairs $\mathcal{Q}(K_3, X_3)\mathcal{S}(X_3, M_3)$ and $\mathcal{Q}(L_3, Y_3)\mathcal{S}(Y_3, M_3)$ have ε-intersecting links and, hence, by our application of Corollary 4.8 to the triples $\mathcal{QUS}$ and $\mathcal{QVS}$ the pairs $\mathcal{U}(K_3, M_3)\mathcal{S}(X_3, M_3)$ and $\mathcal{V}(L_3, M_3)\mathcal{S}(Y_3, M_3)$ have ε-disjoint links, for which reason

$$\Lambda(\mathcal{U}, M_4, k) \not\equiv \Lambda(\mathcal{S}, M_4, x) \quad \text{and} \quad \Lambda(\mathcal{V}, M_4, \ell) \not\equiv \Lambda(\mathcal{S}, M_4, y)\,. \tag{4.24}$$

Moreover, writing $M_4 = \{m, m'\}$ the right part of (4.22) yields

$$|N(S^{xm}, S^{xm'}) \cap N(S^{ym}, S^{ym'})| \geqslant \varepsilon|\mathcal{P}^{mm'}|\,,$$

whence $\Lambda(\mathcal{S}, M_4, x) \equiv \Lambda(\mathcal{S}, M_4, y)$. Together with (4.24) this discloses

$$\Lambda(\mathcal{U}, M_4, k) \equiv \Lambda(\mathcal{V}, M_4, \ell)$$

and, consequently, $\mathcal{U}(K_3, M_3)\mathcal{V}(L_3, M_3)$ has ε-intersecting links, as desired. □

4.7. **Two large disjoint holes.** In this section we establish the existence of two essentially disjoint holes such that most cherries in each hole have a large neighbourhood in the other hole. For that we consider the following sets of unwanted cherries.

Given μ-holes Φ and Ψ on J, $\gamma > 0$, and indices $ijk \in J^{(3)}$ a cherry $(P^{ij}, P^{ik}) \in \mathcal{P}^{ij} \times \mathcal{P}^{ik}$ is *γ-bad* if either

$$\begin{aligned} &(P^{ij}, P^{ik}) \in \Phi^{ij} \times \Phi^{ik} \text{ and } |N(P^{ij}, P^{ik}) \smallsetminus \Psi^{jk}| \geqslant \gamma|\mathcal{P}^{jk}| \\ \text{or} \quad &(P^{ij}, P^{ik}) \in \Psi^{ij} \times \Psi^{ik} \text{ and } |N(P^{ij}, P^{ik}) \smallsetminus \Phi^{jk}| \geqslant \gamma|\mathcal{P}^{jk}|\,. \end{aligned}$$

For $i < j < k$ we denote the sets of all γ-bad *left*, *middle*, and *right* cherries by

$$\mathscr{B}^{ijk}(\Phi, \Psi, \gamma) \subseteq \mathcal{P}^{ij} \times \mathcal{P}^{ik}\,, \quad \mathscr{C}^{ijk}(\Phi, \Psi, \gamma) \subseteq \mathcal{P}^{ij} \times \mathcal{P}^{jk}\,, \text{ and } \mathscr{D}^{ijk}(\Phi, \Psi, \gamma) \subseteq \mathcal{P}^{ik} \times \mathcal{P}^{jk}\,.$$

The following lemma shows that given two disjoint holes Φ and Ψ of width at least $1/3+\varepsilon$ there are either (for a large subset of indices) few γ-bad cherries or there are two other holes covering substantially more space. It might be helpful to point out that eventually we will only use this lemma for $\gamma = \varepsilon/12$.

Lemma 4.17 (density increment lemma). *For every μ, $\varepsilon \geqslant \gamma > 0$ and $t \in \mathbb{N}$ there is $\nu > 0$ such that the following holds. Suppose $\mathcal{A}$ is an ε-wicked reduced hypergraph with index set I and for sufficiently large $J \subseteq I$ we are given ε-disjoint $(\nu, 1/3 + \varepsilon)$-holes Φ and Ψ on J.*

Then, there exists a subset $J_\star \subseteq J$ of size t such that one of the following alternatives occurs.

(*A*) *There exist two ε-disjoint $(\mu, 1/3 + \varepsilon)$-holes $\Phi_\star$ and $\Psi_\star$ on $J_\star$ such that*

$$|\Phi_\star^{ij} \cup \Psi_\star^{ij}| \geqslant |\Phi^{ij} \cup \Psi^{ij}| + \frac{\gamma}{2}|\mathcal{P}^{ij}|$$

for every $ij \in J_\star^{(2)}$

(*B*) *or for all $i < j < k$ in $J_\star$ the sets of γ-bad cherries satisfy*

$$\begin{aligned} |\mathscr{B}^{ijk}(\Phi, \Psi, \gamma)| \leqslant \mu|\mathcal{P}^{ij}||\mathcal{P}^{ik}|\,, \quad |\mathscr{C}^{ijk}(\Phi, \Psi, \gamma)| \leqslant \mu|\mathcal{P}^{ij}||\mathcal{P}^{jk}|\,, \\ \textit{and} \quad |\mathscr{D}^{ijk}(\Phi, \Psi, \gamma)| \leqslant \mu|\mathcal{P}^{ik}||\mathcal{P}^{jk}|\,. \end{aligned}$$

Proof. Given μ, $\varepsilon \geqslant \gamma > 0$ and t we fix auxiliary integers t_1, t_2, t_3, t_4, and we choose ν to satisfy

$$\varepsilon^{-1}, \mu^{-1}, \gamma^{-1}, t \ll t_4 \ll t_3 \ll t_2 \ll t_1, \nu^{-1}\,.$$

Let $\mathcal{A}$, $J \subseteq I$, Φ, and Ψ be as in the statement of the lemma, where J is so large that $|J| \longrightarrow (t_1, t_1, t_1, t)^3_4$. We suppose that (*B*) fails and intend to derive (*A*).

Our assumption on the size of J combined with the failure of *(B)* yields a subset $J_1 \subseteq J$ of size t_1 such that one of the following three statements holds:

(1) $|\mathscr{B}^{ijk}(\Phi, \Psi, \gamma)| > \mu|\mathcal{P}^{ij}||\mathcal{P}^{ik}|$ for all $i < j < k$ in J_1,
(2) $|\mathscr{C}^{ijk}(\Phi, \Psi, \gamma)| > \mu|\mathcal{P}^{ij}||\mathcal{P}^{jk}|$ for all $i < j < k$ in J_1,
(3) or $|\mathscr{D}^{ijk}(\Phi, \Psi, \gamma)| > \mu|\mathcal{P}^{ik}||\mathcal{P}^{jk}|$ for all $i < j < k$ in J_1.

As reversing the order $<$ on I exchanges *(1)* and *(3)*, we may assume that one of the first two cases occurs.

First Case: We have $|\mathscr{B}^{ijk}(\Phi, \Psi, \gamma)| > \mu\,|\mathcal{P}^{ij}||\mathcal{P}^{ik}|$ *for all* $i < j < k$ *in* J_1.

For all $i < j < k$ in J_1 at least one of the sets

$$\mathscr{B}^{ijk}_\Phi = \mathscr{B}^{ijk}(\Phi, \Psi, \gamma) \cap \Phi^{ij} \times \Phi^{ik} \quad \text{and} \quad \mathscr{B}^{ijk}_\Psi = \mathscr{B}^{ijk}(\Phi, \Psi, \gamma) \cap \Psi^{ij} \times \Psi^{ik}$$

must consist of more than $\frac{\mu}{2}|\mathcal{P}^{ij}||\mathcal{P}^{ik}|$ bad cherries. Thus, a further application of Ramsey's theorem allows us to assume that there is a set $J_2 \subseteq J_1$ of size t_2 such that

$$|\mathscr{B}^{ijk}_\Phi| > \frac{\mu}{2}|\mathcal{P}^{ij}||\mathcal{P}^{ik}| \tag{4.25}$$

holds for all $i < j < k$ in J_2.

Claim 4.18. *There are a set* $J_3 \subseteq J_2$ *of size* t_3 *and transversals* $\mathcal{Q}(J_3)$, $\mathcal{R}(J_3)$ *such that*

$$|N(Q^{ij}, R^{ik}) \smallsetminus (\Phi^{jk} \cup \Psi^{jk})| \geqslant \frac{\gamma}{2}|\mathcal{P}^{jk}| \tag{4.26}$$

for all $i < j < k$ *in* J_3 *and*

$$|N(Q^{ik}, Q^{jk}) \cap N(R^{i\ell}, R^{j\ell})| \geqslant \varepsilon|\mathcal{P}^{ij}| \tag{4.27}$$

whenever $i < j < k < \ell$ *are in* J_3.

Proof. Let $\mathcal{A}_2$ be the auxiliary reduced hypergraph with index set J_2 and vertex classes $\mathcal{P}^{ij}$ for $ij \in J_2^{(2)}$ whose constituents are defined by

$$\{P^{ij}, P^{ik}, P^{jk}\} \in E(\mathcal{A}_2^{ijk}) \quad \Longleftrightarrow \quad (P^{ij}, P^{ik}) \in \mathscr{B}^{ijk}_\Phi \smallsetminus \mathscr{L}^{ijk}(\Phi, \gamma/2)$$

for all $i < j < k$ in J_2 and all $(P^{ij}, P^{ik}, P^{jk}) \in \mathcal{P}^{ij} \times \mathcal{P}^{ik} \times \mathcal{P}^{jk}$. Due to (4.1) we have

$$|\mathscr{L}^{ijk}(\Phi, \gamma/2)| \leqslant \frac{2\nu}{\gamma}|\mathcal{P}^{ij}||\mathcal{P}^{ik}| \leqslant \frac{\mu}{4}|\mathcal{P}^{ij}||\mathcal{P}^{ik}|$$

for all $i < j < k$ in J_2 and together with (4.25) this establishes that $\mathcal{A}_2$ is $(\mu/4, \therefore)$-dense.

Together with

$$|\mathscr{L}^{ijk}(\Phi, \gamma/2)| \leqslant \frac{2\nu}{\gamma}|\mathcal{P}^{ij}||\mathcal{P}^{ik}| \qquad \text{and} \qquad |\mathscr{R}^{ijk}(\Phi, \gamma/2)| \leqslant \frac{2\nu}{\gamma}|\mathcal{P}^{ik}||\mathcal{P}^{jk}|\,,$$

and $\nu \ll \gamma, \mu$ this shows that Lemma 3.4 yields a set $J_3 \subseteq J_2$ of size t_3 and transversals $\mathcal{Q}(J_3)$, $\mathcal{R}(J_3)$, and $\mathcal{S}(J_3)$ that avoid $\mathscr{L}(\Phi, \gamma/2)$ and $\mathscr{R}(\Phi, \gamma/2)$ and form an inhabited triple in $\mathcal{A}_2$.

In particular, we have $Q^{ij}R^{ik}S^{jk} \in E(\mathcal{A}_2)$ for all $i < j < k$ in J_3 and thus

$$(Q^{ij}, R^{ik}) \in \mathscr{B}^{ijk}_{\Phi} \smallsetminus \mathscr{L}^{ijk}(\Phi, \gamma/2)\,. \tag{4.28}$$

By the definitions of $\mathscr{B}^{ijk}_{\Phi}$ and $\mathscr{L}^{ijk}(\Phi, \gamma/2)$ this tells us

$$|N(Q^{ij}, R^{ik}) \smallsetminus \Psi^{jk}| \geqslant \gamma|\mathcal{P}^{jk}| \qquad \text{and} \qquad |N(Q^{ij}, R^{ik}) \cap \Phi^{jk}| < \frac{\gamma}{2}|\mathcal{P}^{jk}|\,,$$

and by subtracting these estimates one easily confirms (4.26).

Now let $i < j < k < \ell$ from J_3 be arbitrary. Since $\mathcal{Q}$ and $\mathcal{R}$ avoid $\mathscr{R}(\Phi, \gamma/2)$, we have

$$|N(Q^{ik}, Q^{jk}) \cap \Phi^{ij}| \leqslant \frac{\gamma}{2}|\mathcal{P}^{ij}| < \varepsilon|\mathcal{P}^{ij}| \quad \text{and} \quad |N(R^{i\ell}, R^{j\ell}) \cap \Phi^{ij}| \leqslant \frac{\gamma}{2}|\mathcal{P}^{ij}| < \varepsilon|\mathcal{P}^{ij}|\,.$$

Since each of the three subsets $N(Q^{ik}, Q^{jk})$, $N(R^{i\ell}, R^{j\ell})$, and Φ^{ij} of $\mathcal{P}^{ij}$ has at least the size $(1/3 + \varepsilon)|\mathcal{P}^{ij}|$, this implies (4.27). □

Now Lemma 4.15 applied to J_3 and the transversals $\mathcal{Q}(J_3)$, $\mathcal{R}(J_3)$ yields a set $J_4^+ \subseteq J_3$ of size $t_4 + 1$ such that all $i < j < k < \ell$ in J_4^+ satisfy

$$e\big(N(Q^{ij}, R^{ik}), N(Q^{ij}, R^{i\ell}), N(Q^{ik}, R^{i\ell})\big) \leqslant \nu|\mathcal{P}^{jk}||\mathcal{P}^{j\ell}||\mathcal{P}^{k\ell}|\,.$$

Setting $x = \min(J_4^+)$, $J_4 = J_4^+ \smallsetminus \{x\}$, and

$$\Omega^{jk} = N(Q^{xj}, R^{xk})$$

for all $jk \in J_4^{(2)}$ we obtain

$$e(\Omega^{jk}, \Omega^{j\ell}, \Omega^{k\ell}) \leqslant \nu|\mathcal{P}^{jk}||\mathcal{P}^{j\ell}||\mathcal{P}^{k\ell}|$$

for all $jk\ell \in J_4^{(3)}$. In other words, the set $\Omega = \bigcup_{jk \in J_4^{(2)}} \Omega^{ij}$ is a $(\nu, 1/3 + \varepsilon)$-hole. Moreover, by (4.26) we have

$$\big|\Omega^{jk} \smallsetminus (\Phi^{jk} \cup \Psi^{jk})\big| \geqslant \frac{\gamma}{2}\big|\mathcal{P}^{jk}\big|\,.$$

Now by Corollary 4.14 there exists a subset $J_\star \subseteq J_4$ of size t in which $\Omega \cup \Phi$ and Ψ or $\Omega \cup \Psi$ and Φ are two ε-disjoint μ-holes. Due to (4.26) this shows that *(A)* holds either for $\Phi_\star = \Phi \cup \Omega$ and $\Psi_\star = \Psi$, or for $\Phi_\star = \Phi$ and $\Psi_\star = \Psi \cup \Omega$.

Second Case: We have $|\mathscr{C}^{ijk}(\Phi, \Psi, \gamma)| > \mu\,|\mathcal{P}^{ij}||\mathcal{P}^{jk}|$ for all $i < j < k$ in J_1.
As before we consider the set of ε-bad cherries $\mathscr{C}^{ijk}_{\Phi}$ and $\mathscr{C}^{ijk}_{\Psi}$ restricted to the respective holes and following the same Ramsey argument we find a subset $J_2 \subseteq J$ of size at least t_2 for which we may assume that

$$|\mathscr{C}^{ijk}_{\Phi}| > \frac{\mu}{2}|\mathcal{P}^{ij}||\mathcal{P}^{jk}|$$

holds for all $i < j < k$ in J_2.

Claim 4.19. *There are a set $J_3 \subseteq J_2$ of size t_3 and transversals $\mathcal{Q}(J_3)$, $\mathcal{S}(J_3)$ such that*

$$|N(Q^{ij}, S^{jk}) \smallsetminus (\Phi^{ik} \cup \Psi^{ik})| \geqslant \frac{\gamma}{2}|\mathcal{P}^{ik}| \tag{4.29}$$

whenever $i < j < k$ are in J_3, and

$$|N(Q^{ij}, Q^{ik}) \cap N(S^{j\ell}, S^{k\ell})| \geqslant \varepsilon|\mathcal{P}^{jk}| \quad \textit{and} \quad |N(S^{ik}, S^{i\ell}) \cap N(S^{jk}, S^{j\ell})| \geqslant \varepsilon|\mathcal{P}^{k\ell}| \tag{4.30}$$

for all $i < j < k < \ell$ in J_3.

Proof. This time the constituents of our auxiliary reduced hypergraph $\mathcal{A}_2$ with index set J_2 are defined by

$$\{P^{ij}, P^{ik}, P^{jk}\} \in E(\mathcal{A}_2^{ijk}) \quad \Longleftrightarrow \quad (P^{ij}, P^{jk}) \in \mathscr{C}_\Phi^{ijk} \smallsetminus \mathscr{M}^{ijk}(\Phi, \gamma/2)$$

for $i < j < k$ in J_2 (see Definition 4.2). As in the first case Lemma 3.4 leads to a set $J_3 \subseteq J_2$ of size t_3 and transversals $\mathcal{Q}(J_3)$, $\mathcal{S}(J_3)$ which satisfy $(Q^{ij}, S^{jk}) \in \mathscr{C}_\Phi^{ijk} \smallsetminus \mathscr{M}^{ijk}(\Phi, \gamma/2)$ for all $i < j < k$ in J_3 and avoid the left and right $(\gamma/2)$-exceptional cherries of Φ. Again the first of these properties yields

$$|N(Q^{ij}, S^{jk}) \smallsetminus \Psi^{ik}| \geqslant \gamma|\mathcal{P}^{ik}| \qquad \text{and} \qquad |N(Q^{ij}, S^{jk}) \cap \Phi^{ik}| < \frac{\gamma}{2}|\mathcal{P}^{ik}|\,,$$

and (4.29) follows upon subtraction.

For the proof (4.30) we fix four indices $i < j < k < \ell$ from J_3. The subsets $N(Q^{ij}, Q^{ik})$, $N(S^{j\ell}, S^{k\ell})$, and Φ^{jk} of $\mathcal{P}^{jk}$ have size at least $(1/3+\varepsilon)|\mathcal{P}^{jk}|$ and the third of them intersects the other two in less than $\varepsilon|\mathcal{P}^{jk}|$ vertices. This implies the left part of (4.30). The right side can be shown in the same way, looking at the sets $N(S^{ik}, S^{i\ell})$, $N(S^{jk}, S^{j\ell})$, and $\Phi^{k\ell}$. □

Now we define

$$\Omega_x^{ij} = N(Q^{ix}, S^{xj}) \subseteq \mathcal{P}^{ij}$$

for all $i < x < y$ in J_3. Owing to Lemma 4.16 there exists a set $J_4^+ \subseteq J_3$ of size $2t_4 - 1$ such that

$$e(\Omega_x^{ij}, \Omega_x^{ik}, \Omega_y^{jk}) \leqslant \nu|\mathcal{P}^{ij}||\mathcal{P}^{ik}||\mathcal{P}^{jk}| \tag{4.31}$$

holds for all $i < x < j < y < k$ from J_4^+. Let $J_4^+ = \{j(1), \dots, j(2t_1 - 1)\}$ enumerate the elements of J_4^+ in increasing order, let $J_4 = \{j(1), j(3), \dots, j(2t_4 - 1)\}$ be the t_4-element subset of J_4^+ consisting of the elements occupying odd positions, and set

$$\Omega^{j(2r-1)j(2s-1)} = \Omega_{j(2r)}^{j(2r-1)j(2s-1)} \qquad \text{for all } rs \in [t_4]^{(2)}\,.$$

By (4.31) the set $\Omega = \bigcup_{rs \in [t_4]^{(2)}} \Omega^{j(2r-1)j(2s-1)}$ is a $(\nu, 1/3 + \varepsilon)$-hole on J_4 and in view of (4.29) we can finish as in the first case. □

Now Lemma 4.12 followed by iterative applications of Lemma 4.17 leads to two nonequivalent holes with few bad cherries.

Corollary 4.20. *For every μ, $\varepsilon \geqslant \gamma > 0$ and $t \in \mathbb{N}$ the following holds. If $\mathcal{A}$ is an ε-wicked reduced hypergraph whose index set I is sufficiently large, then there exist a subset $J \subseteq I$*

of size t and ε-disjoint $(\mu, 1/3+\varepsilon)$-holes Φ and Ψ on J such that for all $i<j<k$ in J the sets of γ-bad cherries satisfy

$$|\mathscr{B}^{ijk}(\Phi,\Psi,\gamma)| \leqslant \mu|\mathcal{P}^{ij}||\mathcal{P}^{ik}|\,, \quad |\mathscr{C}^{ijk}(\Phi,\Psi,\gamma)| \leqslant \mu|\mathcal{P}^{ij}||\mathcal{P}^{jk}|\,,$$
$$\textit{and} \qquad |\mathscr{D}^{ijk}(\Phi,\Psi,\gamma)| \leqslant \mu|\mathcal{P}^{ik}||\mathcal{P}^{jk}|\,. \tag{4.32}$$

Proof. By Lemma 4.17 there are functions $f\colon \mathbb{R}_{>0}\times\mathbb{N}\longrightarrow\mathbb{R}_{>0}$ and $g\colon \mathbb{R}_{>0}\times\mathbb{N}\longrightarrow\mathbb{N}$ such that for all $t_\star\in\mathbb{N}$, $\mu_\star\in\mathbb{R}_{>0}$ the conclusion of Lemma 4.17 holds for $\mu_\star$, $t_\star$, $f(\mu_\star,t_\star)$ and $g(\mu_\star,t_\star)$ here in place of μ, t, ν and $|J|$ there.

Starting with $\mu_0=\mu$ and $t_0=t$ we recursively set $\mu_{m+1}=f(\mu_m,t_m)$ and $t_{m+1}=g(\mu_m,t_m)$ for every integer $m\geqslant 0$. Without loss of generality we may assume that the sequence $(\mu_m)_{m\geqslant 0}$ is decreasing and that $t_m\geqslant 2$ for every m. Setting $s=\lceil 4\gamma^{-1}\rceil$ we shall now prove the conclusion of our corollary for $|J|\gg t_s,\mu_s^{-1},\varepsilon^{-1}$.

Due to Lemma 4.12 there are a set $J_s\subseteq J$ of size t_s and two ε-disjoint $(\mu_s,1/3+\varepsilon)$-holes on J_s. Thus there exists a least nonnegative integer $m\leqslant s$ such that there are a set $J_m\subseteq J$ of size t_m and two ε-disjoint $(\mu_m,1/3+\varepsilon)$-holes Φ, Ψ on J_m such that $|\Phi^{ij}|+|\Psi^{ij}|>(s-m)\gamma|\mathcal{P}^{ij}|/2$ holds for every pair $ij\in J_k^{(2)}$.

As our choice of s entails $s\gamma/2\geqslant 2$, we cannot have $m=0$. Thus Lemma 4.17 leads to a set $J_{m-1}\subseteq J_m$ of size t_{m-1} such that either *(A)* or *(B)* holds for μ_{m-1} here in place of μ there. By the minimality of m alternative *(A)* is impossible. For this reason the restrictions of Φ and Ψ to arbitrary t-element subsets of J_{m-1} are as desired. □

4.8. **Bicolourisation.** It remains to argue that by taking a random preimage we can convert Corollary 4.20 into Proposition 2.6.

Proof of Proposition 2.6. Given ε and t we take $\gamma,\mu>0$ and $\ell\in\mathbb{N}$ such that

$$\gamma=\frac{\varepsilon}{12} \qquad \text{and} \qquad \varepsilon,t^{-1}\gg\ell^{-1}\gg\mu$$

and consider an ε-wicked reduced hypergraph $\mathcal{A}$ whose index set I is sufficiently large. Due to Corollary 4.20 there are a set $J\subseteq I$ of size t and ε-disjoint μ-holes Φ, Ψ on J such that for all $i<j<k$ in J we have

$$|\mathscr{B}^{ijk}(\Phi,\Psi,\gamma)| \leqslant \mu|\mathcal{P}^{ij}||\mathcal{P}^{ik}|\,, \quad |\mathscr{C}^{ijk}(\Phi,\Psi,\gamma)| \leqslant \mu|\mathcal{P}^{ij}||\mathcal{P}^{jk}|\,,$$
$$\text{and} \quad |\mathscr{D}^{ijk}(\Phi,\Psi,\gamma)| \leqslant \mu|\mathcal{P}^{ik}||\mathcal{P}^{jk}|\,. \tag{4.33}$$

Next we define a reduced subhypergraph $\mathcal{A}_1$ of $\mathcal{A}$ admitting a bicolouring φ_1 which satisfies, with only few exceptions, the minimum codegree condition $\tau_2(\mathcal{A}_1,\varphi_1)\geqslant 1/3+\varepsilon/4$. To this end we consider for every pair $ij\in J^{(2)}$ the sets

$$\mathfrak{R}^{ij}=\Phi^{ij}\smallsetminus\Psi^{ij} \quad \text{and} \quad \mathfrak{B}^{ij}=\Psi^{ij}\smallsetminus\Phi^{ij}\,,$$

and then we set $\mathfrak{R}=\bigcup_{ij\in J^{(2)}}\mathfrak{R}^{ij}$ as well as $\mathfrak{B}=\bigcup_{ij\in J^{(2)}}\mathfrak{B}^{ij}$. Now let $\mathcal{A}_1$ be the reduced hypergraph with index set J, vertex classes $\mathcal{P}_1^{ij}=\mathfrak{R}^{ij}\,\dot\cup\,\mathfrak{B}^{ij}\subseteq\mathcal{P}^{ij}$ for every $ij\in J^{(2)}$, and edges

$$E(\mathcal{A}_1) = E(\mathfrak{R} \cup \mathfrak{B}) \smallsetminus \big(E(\mathfrak{R}) \cup E(\mathfrak{B})\big)\,.$$

Since Φ and Ψ are ε-disjoint and have width at least $1/3 + \varepsilon$ we have

$$\big|\mathcal{P}_1^{ij}\big| \geqslant \Big(\frac{2}{3} + \varepsilon\Big)\big|\mathcal{P}^{ij}\big| \tag{4.34}$$

for every $ij \in J^{(2)}$.

It is plain that the map $\varphi_1 \colon V(\mathcal{A}_1) \longrightarrow \{\text{red}, \text{blue}\}$ defined by $\varphi_1^{-1}(\text{red}) = \mathfrak{R}$ and $\varphi_1^{-1}(\text{blue}) = \mathfrak{B}$ is a bicolouring of $\mathcal{A}_1$.

Claim 4.21. *In $\mathcal{A}_1$ all monochromatic cherries (P_1^{ij}, P_1^{ik}) that fail to be γ-bad in $\mathcal{A}$ have codegree at least $(1/3 + \varepsilon/4)|\mathcal{P}_1^{jk}|$.*

Proof. Suppose $i < j < k$ and that $(P_1^{ij}, P_1^{ik}) = (R^{ij}, R^{ik}) \in \mathfrak{R}^{ij} \times \mathfrak{R}^{ik}$ is a red left cherry not belonging to $\mathscr{B}^{ijk}(\Phi, \Psi, \gamma)$. Due to

$$|N_{\mathcal{A}}(R^{ij}, R^{ik}) \smallsetminus \Psi^{jk}| \leqslant \gamma|\mathcal{P}^{jk}|$$

we have

$$\begin{aligned}
|N_{\mathcal{A}_1}(R^{ij}, R^{ik})| &= |N_{\mathcal{A}}(R^{ij}, R^{ik}) \cap \mathfrak{B}^{jk}| \\
&\geqslant |N_{\mathcal{A}}(R^{ij}, R^{ik})| - |N_{\mathcal{A}}(R^{ij}, R^{ik}) \smallsetminus \Psi^{jk}| - |\Phi^{jk} \cap \Psi^{jk}| \\
&\geqslant \left(\frac{1}{3} + \varepsilon\right)|\mathcal{P}^{jk}| - \gamma|\mathcal{P}^{jk}| - |\Phi^{jk} \cap \Psi^{jk}| \\
&\geqslant \left(\frac{1}{3} + \frac{\varepsilon}{4}\right)\big(|\mathcal{P}^{jk}| - |\Phi^{jk} \cap \Psi^{jk}|\big) + \frac{2}{3}\big(\varepsilon|\mathcal{P}^{jk}| - |\Phi^{jk} \cap \Psi^{jk}|\big) \\
&\geqslant \left(\frac{1}{3} + \frac{\varepsilon}{4}\right)|\mathcal{P}_1^{jk}|\,,
\end{aligned}$$

where the penultimate inequality uses the definition of γ and the last inequality follows from $\mathcal{P}_1^{jk} \subseteq \mathcal{P}^{jk} \smallsetminus (\Phi^{jk} \cap \Psi^{jk})$. This concludes the proof for red left cherries and all other cases can be treated analogously. □

Similar as in [10, Lemma 4.2] we will define the reduced hypergraph $\mathcal{A}_\star$ by taking the preimage of a random homomorphism $h \in \mathfrak{A}(\mathcal{A}_1, \ell)$. Recall from Definition 3.5 that for every map $h \in \mathfrak{A}(\mathcal{A}_1, \ell)$ the associated reduced hypergraph $\mathcal{A}_h$ has index set J and vertex classes $\mathcal{P}_\bullet^{ij}$ of size ℓ.

Observe that there is no $h \in \mathfrak{A}(\mathcal{A}_1, \ell)$ such that $\mathcal{A}_h$ supports a $K_5^{(3)}$, because otherwise the homomorphism h would show that $\mathcal{A}_1 \subseteq \mathcal{A}$ supports a $K_5^{(3)}$ as well, contrary to $\mathcal{A}$ being wicked. Furthermore, for every $h \in \mathfrak{A}(\mathcal{A}_1, \ell)$ the map $\varphi_h = \varphi_1 \circ h$ is a bicolouring of $\mathcal{A}_h$. So it remains to show that if h gets chosen uniformly at random, then with positive probability the event

$$\tau_2(\mathcal{A}_h, \varphi_h) \geqslant \frac{1}{3} + \frac{\varepsilon}{8}$$

occurs. We estimate for each cherry of $\mathcal{A}_h$ the probability that it violates this condition.

Claim 4.22. *If $ijk \in J^{(3)}$ and $(P^{ij}_\bullet, P^{ik}_\bullet) \in \mathcal{P}^{ij}_\bullet \times \mathcal{P}^{ik}_\bullet$ is a cherry of $\mathcal{A}_h$, then the event $\mathcal{X}$ that*

$$\varphi_h(P^{ij}_\bullet) = \varphi_h(P^{ik}_\bullet) \quad \textit{and} \quad |N_{\mathcal{A}_h}(P^{ij}_\bullet, P^{ik}_\bullet)| < \Big(\frac{1}{3} + \frac{\varepsilon}{8}\Big)|\mathcal{P}^{jk}_\bullet|$$

has at most the probability $3\mu + \exp\big(-\frac{\varepsilon^2\ell}{128}\big)$.

Proof. Without loss of generality we may assume $i < j < k$. By the law of total probability we have

$$\mathbb{P}(\mathcal{X}) = \frac{1}{|\mathcal{P}^{ij}_1||\mathcal{P}^{ik}_1|} \sum_{(P^{ij}, P^{ik}) \in \mathcal{P}^{ij}_1 \times \mathcal{P}^{ik}_1} \mathbb{P}\left(\mathcal{X} \,\middle|\, h(P^{ij}_\bullet) = P^{ij} \text{ and } h(P^{ik}_\bullet) = P^{ik}\right) .$$

Note that cherries (P^{ij}, P^{ik}) consisting of two vertices with different colours contribute zero to this sum. Moreover, due to (4.33) the total contribution from cherries (P^{ij}, P^{ik}) belonging to $\mathscr{B}^{ijk}(\Phi, \Psi, \gamma)$ is at most

$$\frac{\mu|\mathcal{P}^{ij}||\mathcal{P}^{ik}|}{|\mathcal{P}^{ij}_1||\mathcal{P}^{ik}_1|} \overset{(4.34)}{\leqslant} \left(\frac{3}{2}\right)^2 \mu < 3\mu\,.$$

Furthermore, for P^{ij}, P^{ik} of the same colour with $(P^{ij}, P^{ik}) \notin \mathscr{B}^{ijk}(\Phi, \Psi, \gamma)$ Claim 4.21 combined with Chernoff's inequality tells us

$$\mathbb{P}\left(\mathcal{X} \,\middle|\, h(P^{ij}_\bullet) = P^{ij} \text{ and } h(P^{ik}_\bullet) = P^{ik}\right) \leqslant \exp\left(-\tfrac{\varepsilon^2\ell}{128}\right)$$

and Claim 4.22 follows. □

Since $\mathcal{A}_h$ has $3\ell^2\binom{t}{3}$ cherries, Claim 4.22 implies

$$\mathbb{P}\left(\tau_2(\mathcal{A}_h, \varphi_h) < \frac{1}{3} + \frac{\varepsilon}{8}\right) \leqslant 3\ell^2\binom{t}{3}\left(3\mu + \exp\left(-\tfrac{\varepsilon^2\ell}{128}\right)\right) .$$

Owing to the hierarchy $\mu \ll \ell^{-1} \ll t^{-1}$ this probability is smaller than 1 and, therefore, there is a map $h \in \mathfrak{A}(\mathcal{A}_1, \ell)$ for which $\mathcal{A}_h$ has the desired properties. □

§5. Cliques on five vertices in bicoloured reduced hypergraphs

In this section we establish Proposition 2.7 and show that bicoloured reduced hypergraphs with minimum monochromatic codegree density bigger than 1/3 support a $K^{(3)}_5$.

In the proof we shall use the following types of neighbourhoods in reduced hypergraphs $\mathcal{A}$. For two vertices P, $P' \in V(\mathcal{A})$ and a subset $\mathcal{U} \subseteq V(\mathcal{A})$ we denote by $N_{\mathcal{U}}(P, P')$ the neighbourhood restricted to $\mathcal{U}$. Similarly, for two subsets $\mathcal{U}$, $\mathcal{U}' \subseteq V(\mathcal{A})$ we write $N_{\mathcal{U}\times\mathcal{U}'}(P)$ for the set of pairs in $\mathcal{U} \times \mathcal{U}'$ that together with P form a hyperedge in $\mathcal{A}$, i.e.,

$$N_{\mathcal{U}}(P, P') = \{U \in \mathcal{U} \colon PP'U \in E(\mathcal{A})\}$$
$$\text{and} \quad N_{\mathcal{U}\times\mathcal{U}'}(P) = \{(U, U') \in \mathcal{U} \times \mathcal{U}' \colon PUU' \in E(\mathcal{A})\}\,.$$

Proof of Proposition 2.7. Clearly we may assume that $\varepsilon < \frac{1}{6}$. Fix a sufficiently small auxiliary constant ξ with $0 < \xi \ll \varepsilon$ such that $\frac{1/6-\varepsilon}{\xi}$ is equal to some positive integer s. Moreover, let I be a sufficiently large index set such that its cardinality satisfies the partition relation $|I| \longrightarrow (5)^2_s$, meaning that it is at least as large as the s-colour Ramsey number for the graph clique K_5. Let $\mathcal{A}$ be a bicoloured reduced hypergraph with index set I and vertex classes $\mathcal{P}^{ij}$ for $ij \in I^{(2)}$ and let the bicolouring $\varphi\colon V(\mathcal{A}) \longrightarrow \{\text{red}, \text{blue}\}$ satisfy $\tau_2(\mathcal{A}, \varphi) \geqslant 1/3 + \varepsilon$.

For every $ij \in I^{(2)}$ we set

$$\mathfrak{R}^{ij} = \varphi^{-1}(\text{red}) \cap \mathcal{P}^{ij} \qquad \text{and} \qquad \varrho_{ij} = \frac{|\mathfrak{R}^{ij}|}{|\mathcal{P}^{ij}|}$$

and, analogously, we define $\mathfrak{B}^{ij} = \varphi^{-1}(\text{blue}) \cap \mathcal{P}^{ij}$ and $\beta_{ij} = |\mathfrak{B}^{ij}|/|\mathcal{P}^{ij}|$. In view of (2.1), the assumption on $\tau_2(\mathcal{A}, \varphi)$ implies that all ϱ_{ij}, β_{ij} are in $[1/3 + \varepsilon, 2/3 - \varepsilon]$. Splitting this interval into s intervals of length 2ξ, the size of I yields a subset $J \subseteq I$ of size 5 such that all β_{ij} with $ij \in J^{(2)}$ are in the same interval. Let β be the centre of this interval and set $\varrho = 1 - \beta$. We thus arrive at

$$\beta_{ij} = \beta \pm \xi \qquad \text{and} \qquad \varrho_{ij} = \varrho \pm \xi$$

for all $ij \in J^{(2)}$. Without loss of generality we may assume $\beta \leqslant \varrho$, which implies

$$\frac{1}{3} + \varepsilon \leqslant \beta - \xi < \beta \leqslant \frac{1}{2} \leqslant \varrho < \varrho + \xi \leqslant \frac{2}{3} - \varepsilon\,. \tag{5.1}$$

For $ijk \in J^{(3)}$ the codegree condition translates for red vertices $R^{ij} \in \mathfrak{R}^{ij}$ and $R^{ik} \in \mathfrak{R}^{ik}$ to

$$\begin{aligned} |N_{\mathfrak{B}^{jk}}(R^{ij}, R^{ik})| = d(R^{ij}, R^{jk}) &\geqslant \left(\frac{1}{3} + \varepsilon\right) |\mathcal{P}^{jk}| \\ &\geqslant \left(\frac{1}{3} + \varepsilon\right) \left(\frac{1}{\beta + \xi}\right) |\mathfrak{B}^{jk}| \geqslant \left(\frac{1}{3\beta} + \frac{\varepsilon}{2}\right) |\mathfrak{B}^{jk}|\,, \end{aligned} \tag{5.2}$$

where we used $\xi \ll \varepsilon, \beta$ for the last inequality. Similarly, for blue vertices we have

$$|N_{\mathfrak{R}^{jk}}(B^{ij}, B^{ik})| \geqslant \left(\frac{1}{3\varrho} + \frac{\varepsilon}{2}\right) |\mathfrak{B}^{jk}|\,. \tag{5.3}$$

We may rename the indices in J and assume that $J = \mathbb{Z}/5\mathbb{Z}$. We shall show that $\mathcal{A}$ restricted to J supports a $K_5^{(3)}$. For that we have to find ten vertices $P^{ij} \in \mathcal{P}^{ij}$, one for every $ij \in J^{(2)}$, such that for all of the ten triples $ijk \in J^{(3)}$ the vertices P^{ij}, P^{ik}, and P^{jk} span a hyperedge in the constituent $\mathcal{A}^{ijk}$. For every $i \in J = \mathbb{Z}/5\mathbb{Z}$ we will select $P^{i,i+1}$ from $\mathfrak{B}^{i,i+1}$ and $P^{i,i+2}$ from $\mathfrak{R}^{i,i+2}$. Since $\mathcal{A}$ contains no monochromatic triples as hyperedges, this choice for the colour classes is up to a permutation of indices unavoidable, as it corresponds to the unique 2-colouring of $E(K_5)$ with no monochromatic triangle.

The rest of the proof is based on several averaging arguments relying on the minimum degree condition. For generic vertices from $\mathfrak{R}$ and $\mathfrak{B}$ we shall use capital letters R and B.

In the process we will make appropriate choices to fix the ten special vertices that induce the supported $K_5^{(3)}$. For those vertices we will use small letters r and b depending on their colour.

We begin with the selection of $r^{14} \in \mathfrak{R}^{14}$. Applying (5.3) to all pairs of vertices $B^{15} \in \mathfrak{B}^{15}$ and $B^{45} \in \mathfrak{B}^{45}$ implies that the total number of hyperedges in $\mathcal{A}^{145}$ crossing the sets $\mathfrak{R}^{14}$, $\mathfrak{B}^{15}$, and $\mathfrak{B}^{45}$ is at least

$$|\mathfrak{B}^{15}||\mathfrak{B}^{45}| \cdot \left(\frac{1}{3\varrho} + \frac{\varepsilon}{2}\right) |\mathfrak{R}^{14}|\,.$$

Consequently, we can fix some vertex $r^{14} \in \mathfrak{R}^{14}$ such that

$$|N_{\mathfrak{B}^{15}\times\mathfrak{B}^{45}}(r^{14})| \geqslant \left(\frac{1}{3\varrho} + \frac{\varepsilon}{2}\right) |\mathfrak{B}^{15}||\mathfrak{B}^{45}|\,. \tag{5.4}$$

The following claim fixes the four vertices b^{12}, b^{34} and r^{13}, r^{24}.

Claim 5.1. *There are blue vertices $b^{12} \in \mathfrak{B}^{12}$, $b^{34} \in \mathfrak{B}^{34}$ and red vertices $r^{13} \in \mathfrak{R}^{13}$, $r^{24} \in \mathfrak{R}^{24}$ such that*

- *(i) $b^{12}r^{14}r^{24}$ and $r^{13}r^{14}b^{34}$ are hyperedges in $\mathcal{A}$*
- *(ii) and $|N_{\mathfrak{B}^{23}}(b^{12}, r^{13}) \cap N_{\mathfrak{B}^{23}}(r^{24}, b^{34})| \geqslant \big(1 - \frac{1}{3\beta}\big)|\mathfrak{B}^{23}|$.*

Proof. Owing to (5.2) for every $R^{13} \in \mathfrak{R}^{13}$ we have $d(R^{13}, r^{14}) \geqslant \big(\frac{1}{3\beta} + \frac{\varepsilon}{2}\big)|\mathfrak{B}^{34}|$ and, hence, there is a vertex $b^{34} \in \mathfrak{B}^{34}$ such that

$$|N_{\mathfrak{R}^{13}}(r^{14}, b^{34})| \geqslant \left(\frac{1}{3\beta} + \frac{\varepsilon}{2}\right) |\mathfrak{R}^{13}| \geqslant \frac{\varrho}{3\beta}|\mathcal{P}^{13}|\,. \tag{5.5}$$

Similarly, we can fix a vertex $r^{24} \in \mathfrak{R}^{24}$ such that

$$|N_{\mathfrak{B}^{23}}(r^{24}, b^{34})| \geqslant \frac{1}{3\varrho}|\mathfrak{B}^{23}|\,. \tag{5.6}$$

Recalling that $|\mathfrak{R}^{13}| \leqslant (\varrho + \xi)|P^{13}|$ for every $B^{12} \in \mathfrak{B}^{12}$ and $B^{23} \in \mathfrak{B}^{23}$ we have

$$\begin{aligned}
\big|N_{\mathfrak{R}^{13}}(B^{12}, B^{23}) \cap N_{\mathfrak{R}^{13}}(r^{14}, b^{34})\big| &\geqslant \left(\frac{1}{3} + \varepsilon\right) |\mathcal{P}^{13}| + \big|N_{\mathfrak{R}^{13}}(r^{14}, b^{34})\big| - |\mathfrak{R}^{13}| \\
&\geqslant \big|N_{\mathfrak{R}^{13}}(r^{14}, b^{34})\big| - \left(\varrho + \xi - \frac{1}{3} - \varepsilon\right) |\mathcal{P}^{13}| \\
&\overset{(5.5)}{\geqslant} \left(1 - 3\beta + \frac{\beta}{\varrho}\right) \big|N_{\mathfrak{R}^{13}}(r^{14}, b^{34})\big| \\
&\geqslant \left(3\varrho - \frac{\varrho}{\beta}\right) \big|N_{\mathfrak{R}^{13}}(r^{14}, b^{34})\big|\,,
\end{aligned}$$

where the last estimate uses $\beta + \varrho = 1$ and $\frac{\beta}{\varrho} + \frac{\varrho}{\beta} \geqslant 2$. Hence, the number of hyperedges crossing $N_{\mathfrak{B}^{12}}(r^{14}, r^{24})$, $N_{\mathfrak{B}^{23}}(r^{24}, b^{34})$, and $N_{\mathfrak{R}^{13}}(r^{14}, b^{34})$ is at least

$$|N_{\mathfrak{B}^{12}}(r^{14}, r^{24})||N_{\mathfrak{B}^{23}}(r^{24}, b^{34})| \cdot \left(3\varrho - \frac{\varrho}{\beta}\right) |N_{\mathfrak{R}^{13}}(r^{14}, b^{34})|\,.$$

Consequently, there exist $b^{12} \in N_{\mathfrak{B}^{12}}(r^{14}, r^{24})$ and $r^{13} \in N_{\mathfrak{R}^{13}}(r^{14}, b^{34})$ such that

$$\begin{aligned} |N_{\mathfrak{B}^{23}}(b^{12}, r^{13}) \cap N_{\mathfrak{B}^{23}}(r^{24}, b^{34})| &\geqslant \left(3\varrho - \frac{\varrho}{\beta}\right) |N_{\mathfrak{B}^{23}}(r^{24}, b^{34})| \\ &\overset{(5.6)}{\geqslant} \left(1 - \frac{1}{3\beta}\right) |\mathfrak{B}^{23}|\,. \end{aligned}$$

□

The next claim fixes the four vertices b^{15}, b^{45} and r^{25}, r^{35}. Together with Claim 5.1 this fixes all vertices except b^{23} and both claims together guarantee those seven hyperedges supporting a $K_5^{(3)}$ that do not involve b^{23}.

Claim 5.2. *There exist blue vertices $b^{15} \in \mathfrak{B}^{15}$, $b^{45} \in \mathfrak{B}^{45}$ and red vertices $r^{25} \in \mathfrak{R}^{25}$, $r^{35} \in \mathfrak{R}^{35}$ such that $b^{12}b^{15}r^{25}$, $r^{13}b^{15}r^{35}$, $r^{14}b^{15}b^{45}$, $r^{24}r^{25}b^{45}$, and $b^{34}r^{35}b^{45}$ are hyperedges in $\mathcal{A}$.*

Proof. We consider two sets of "candidates" for the pair (b^{15}, b^{45}) that are relevant for the existence of r^{25} and r^{35}. More precisely, we set

$$\begin{aligned} G_1 &= \{(B^{15}, B^{45}) \in \mathfrak{B}^{15} \times \mathfrak{B}^{45} \colon N_{\mathfrak{R}^{25}}(b^{12}, B^{15}) \cap N_{\mathfrak{R}^{25}}(r^{24}, B^{45}) \neq \varnothing\} \\ \text{and} \quad G_2 &= \{(B^{15}, B^{45}) \in \mathfrak{B}^{15} \times \mathfrak{B}^{45} \colon N_{\mathfrak{R}^{35}}(r^{13}, B^{15}) \cap N_{\mathfrak{R}^{35}}(b^{34}, B^{45}) \neq \varnothing\}\,. \end{aligned}$$

Note that for every $B^{15} \in \mathfrak{B}^{15}$ there is some $R^{25} \in N_{\mathfrak{R}^{25}}(b^{12}, B^{15})$ and we have

$$|N_{\mathfrak{B}^{45}}(r^{24}, R^{25})| \overset{(5.2)}{\geqslant} \frac{1}{3\beta} |\mathfrak{B}^{45}|\,.$$

Clearly, $\{B^{15}\} \times N_{\mathfrak{B}^{45}}(r^{24}, R^{25}) \subseteq G_1$ and, hence, we establish

$$|G_1| \geqslant \frac{1}{3\beta} |\mathfrak{B}^{15}||\mathfrak{B}^{45}|\,. \tag{5.7}$$

A symmetric argument yields the same bound for G_2. Combining (5.7) and the same bound for G_2 with (5.4) leads to

$$|G_1| + |G_2| + |N_{\mathfrak{B}^{15} \times \mathfrak{B}^{45}}(r^{14})| \geqslant \left(\frac{2}{3\beta} + \frac{1}{3\varrho} + \frac{\varepsilon}{2}\right) |\mathfrak{B}^{15}||\mathfrak{B}^{45}| \overset{(5.1)}{>} 2\,|\mathfrak{B}^{15}||\mathfrak{B}^{45}|\,.$$

Consequently, we can fix a pair $(b^{15}, b^{45}) \in G_1 \cap G_2 \cap N_{\mathfrak{B}^{15} \times \mathfrak{B}^{45}}(r^{14})$. Moreover, having fixed b^{15} and b^{45} this yields a vertex $r^{25} \in \mathfrak{R}^{25}$ from the non-empty intersection considered in the definition of G_1. Similarly, G_2 leads to our choice of $r^{35} \in \mathfrak{R}^{35}$.

Since $(b^{15}, b^{45}) \in N_{\mathfrak{B}^{15} \times \mathfrak{B}^{45}}(r^{14})$, the hyperedge $r^{14}b^{15}b^{45}$ exists in $\mathcal{A}$ and the other four hyperedges result from the definitions of G_1 and G_2. □

As mentioned above, Claims 5.1 and 5.2 fix all vertices except $b^{23} \in \mathfrak{B}^{23}$ and all hyperedges not involving b^{23}. For the three remaining hyperedges it suffices to show that

$$N_{\mathfrak{B}^{23}}(b^{12}, r^{13}) \cap N_{\mathfrak{B}^{23}}(r^{24}, b^{34}) \cap N_{\mathfrak{B}^{23}}(r^{25}, r^{35}) \neq \varnothing\,.$$

Claim 5.1 *(ii)* and (5.2) imply

$$\begin{aligned}\big|N_{\mathfrak{B}^{23}}(b^{12},r^{13}) \cap N_{\mathfrak{B}^{23}}(r^{24},b^{34}) \cap N_{\mathfrak{B}^{23}}(r^{25},r^{35})\big| \\ \geqslant \big|N_{\mathfrak{B}^{23}}(b^{12},r^{13}) \cap N_{\mathfrak{B}^{23}}(r^{24},b^{34})\big| + \big|N_{\mathfrak{B}^{23}}(r^{25},r^{35})\big| - \big|\mathfrak{B}^{23}\big| \\ \overset{(5.2)}{\geqslant} \Big(1 - \frac{1}{3\beta} + \frac{1}{3\beta} + \frac{\varepsilon}{2} - 1\Big)\,|\mathfrak{B}^{23}| > 0\,.\end{aligned}$$

Hence a suitable choice for b^{23} exists and, therefore, $\mathcal{A}$ restricted to J supports a $K_5^{(3)}$. □

§6. Concluding Remarks

We close with a few related open problems and possible directions for future research.

6.1. **Turán problems for cliques in $\boldsymbol{\wedge}$-dense hypergraphs.** In view of Theorems 1.2 and 1.3 for cliques $K_\ell^{(3)}$ with $\ell \leqslant 16$ vertices only the cases $\ell = 9$ and 10 are still unresolved and closing the bounds

$$\frac{1}{2} \leqslant \pi_{\wedge}(K_9^{(3)}) \leqslant \pi_{\wedge}(K_{10}^{(3)}) \leqslant \frac{2}{3}$$

would be interesting. It seems plausible that by combining our main result with the ideas in [14] one can derive the improved upper bound $\pi_{\wedge}(K_{10}^{(3)}) \leqslant \frac{3}{5}$. More generally, it seems that $\pi_{\wedge}(K_r^{(3)}) = \alpha$ implies $\pi_{\wedge}(K_{2r}^{(3)}) \leqslant \frac{1}{2-\alpha}$ and we shall return to this topic in the near future.

Determining the value $\pi_{\wedge}(K_\ell^{(3)})$ for large values of ℓ might be a challenging problem and one may first focus on the asymptotic behaviour. For every $\ell \geqslant 3$ Theorem 1.2 tells us

$$\pi_{\wedge}(K_\ell^{(3)}) \leqslant 1 - \frac{1}{\log_2(\ell)}\,. \tag{6.1}$$

For a lower bound we consider the following well known random construction.

Example 6.1. *For $r \geqslant 2$ we consider random hypergraphs $H_\varphi = (V, E_\varphi)$ with the edge set defined by the non-monochormatic triangles of a random r-colouring $\varphi\colon V^{(2)} \longrightarrow [r]$ for a sufficiently large vertex set V. It is easy to check that for any fixed $\eta > 0$ with high probability such hypergraphs H_φ are $(\eta, \frac{r-1}{r}, \wedge)$-dense. On the other hand, if ℓ is at least as large as $R(3;r)$, the r-colour Ramsey number for graph triangles, then every such H_φ is $K_\ell^{(3)}$-free.*

Consequently, Example 6.1 yields

$$\pi_{\wedge}(K_\ell^{(3)}) \geqslant 1 - \frac{1}{r}\,, \text{ whenever } \ell \geqslant R(3;r)$$

and using the simple upper bound $R(3;r) \leqslant 3\,r!$ we arrive at

$$\pi_{\wedge}(K_\ell^{(3)}) \geqslant 1 - \frac{\log_2\log_2(\ell)}{\log_2(\ell)} \tag{6.2}$$

for sufficiently large ℓ. Comparing the bounds in (6.1) and (6.2) leads to the following problem.

Problem 6.2. *Determine the asymptotic behaviour of* $1 - \pi_{\wedge}(K_\ell^{(3)})$.

6.2. **Turán problems for hypergraphs with uniformly dense links.** As discussed in the introduction there is a small difference between Theorem 1.3 and Corollary 1.5. Below we briefly elaborate on these differences.

In this work we study $\wedge$-dense hypergraphs, which are defined by the lower bound condition (1.1) in Definition 1.1. Requiring in addition a matching upper bound, i.e., replacing (1.1) by

$$\big|e_{\wedge}(P,Q) - d\,|\mathcal{K}_{\wedge}(P,Q)|\big| \leqslant \eta |V|^3\,,$$

leads to the notion of $(\eta, d, \wedge)$-*quasirandom* hypergraphs. Clearly, we can transfer the definition of $\pi_{\wedge}(F)$ in (1.2) and define the Turán-density $\pi'_{\wedge}(F)$ by restricting to $\wedge$-quasirandom hypergraphs H

$$\pi'_{\wedge}(F) = \sup\{d \in [0,1] \colon \text{for every } \eta > 0 \text{ and } n \in \mathbb{N} \text{ there exists an } F\text{-free, } (\eta, d, \wedge)\text{-quasirandom hypergraph with at least } n \text{ vertices}\}\,.$$

By definition we have $\pi'_{\wedge}(F) \leqslant \pi_{\wedge}(F)$ for every hypergraph F and one may wonder if this inequality is sometimes strict.

For $K_5^{(3)}$ it is easy to check that the lower bound construction in Example 1.4 yields $K_5^{(3)}$-free $(\eta, 1/3, \wedge)$-quasirandom hypergraphs for every fixed $\eta > 0$ and, hence,

$$\pi'_{\wedge}(K_5^{(3)}) = \pi_{\wedge}(K_5^{(3)}) = \frac{1}{3}\,.$$

On the other hand, the lower bound construction for $K_6^{(3)}$ from [14] is given by Example 6.1 for $r = 2$. In those hypergraphs H_φ we can take P and Q to be the pairs in colour 1 and 2 respectively and get

$$e_{\wedge}(P,Q) = |\mathcal{K}_{\wedge}(P,Q)|\,,$$

i.e., they have relative density 1. Therefore, the hypergraphs H_φ are only $(\eta, 1/2, \wedge)$-dense, but not $(\eta, 1/2, \wedge)$-quasirandom. In fact, we are not aware of any matching quasirandom lower bound construction for $\pi_{\wedge}(K_6^{(3)})$ and it seems possible that $\pi'_{\wedge}(K_6^{(3)})$ is strictly smaller than $\pi_{\wedge}(K_6^{(3)})$ suggesting the following general problem.*

Problem 6.3. *Which hypergraphs F satisfy $\pi'_{\wedge}(F) < \pi_{\wedge}(F)$?*

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*We remark that for the concepts of $\therefore$-dense/quasirandom hypergraphs there is no difference for the corresponding Turán-densities, as every $\therefore$-dense hypergraph contains large $\therefore$-quasirandom hypergraphs of at least the same density.

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